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Finite-dimensional irreducible modules for the three-point $\mathfrak{sl}_2$ loop algebra

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Abstract

Recently Brian Hartwig and the second author found a presentation for the three-point $\mathfrak{sl}_2$ loop algebra by generators and relations. To obtain this presentation they defined a Lie algebra $\boxtimes$ by generators and relations, and displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. In this paper we describe the finite-dimensional irreducible $\boxtimes$-modules from multiple points of view.

Keywords. Tetrahedron Lie algebra, tridiagonal pair, Onsager Lie algebra, Kac-Moody algebra.
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1 Introduction

In [15] Hartwig and the second author found a presentation for the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra $\boxtimes$ (pronounced "tet") by generators and relations, and displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. $\boxtimes$ has essentially six generators, and it is natural to identify these with the six edges of a tetrahedron. The action of the symmetric group S_4 on the tetrahedron induces an action of S_4 on $\boxtimes$ as a group of automorphisms [15, Section 2]. For each face of the tetrahedron the three surrounding edges form a basis for a subalgebra of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$ [15, Corollary 12.4]. Any five of the six edges of the tetrahedron generate a subalgebra of $\boxtimes$ that is isomorphic to the $\mathfrak{sl}_2$ loop algebra [15, Corollary 12.6]. Each pair of opposite edges of the tetrahedron generate a subalgebra of $\boxtimes$ that is isomorphic to the Onsager algebra $\mathcal{O}$ [15, Corollary 12.5]; let us call these Onsager subalgebras. Then $\boxtimes$ is the direct sum of its three Onsager subalgebras [15, Theorem 11.6]. In [12] Elduque found an attractive decomposition of $\boxtimes$ into a direct sum of three abelian subalgebras, and he showed how these subalgebras are related to the Onsager subalgebras. In [25] Pascasio and the second author gave an action of $\boxtimes$ on the standard module for each Hamming association scheme. In [3] Bremner obtained the universal central extension of the three-point

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$\mathfrak{sl}_2$ loop algebra. By modifying the defining relations for $\boxtimes$, Benkart and the second author obtained a presentation for this extension by generators and relations [1]. In [14] Hartwig classified the finite-dimensional irreducible $\boxtimes$-modules over an algebraically closed field $\mathbb{F}$ with characteristic 0. He did this by displaying a bijection between (i) the set of isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules; (ii) the set of isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules that have type $(0,0)$ [14, Theorems 1.7, 1.8]. The modules in (ii) were classified earlier by Davies [9], [10]; see also Date and Roan [8]. A summary of this classification is given in [14, Theorems 1.3, 1.4, 1.6].

In this paper we consider further the finite-dimensional irreducible $\boxtimes$-modules. We pick up where Davies, Date and Roan, and Hartwig left off, and obtain a description of these modules that has more detail and takes a different point of view. Our results are summarized as follows. Among the finite-dimensional irreducible $\boxtimes$-modules we identify a special case called an evaluation module. Each evaluation module is written $V_d(a)$, where $d \geq 1$ is an integer and $a \in \mathbb{F}\backslash\{0,1\}$. The scalar a is called the evaluation parameter. To get the $\boxtimes$-module $V_d(a)$ we start with the irreducible $\mathfrak{sl}_2$-module V_d of dimension $d+1$, and pull back the $\mathfrak{sl}_2$-module structure via an evaluation homomorphism $EV_a : \boxtimes \to \mathfrak{sl}_2$. We note that our evaluation homomorphism is different from the one in [8]. We show that every finite-dimensional irreducible $\boxtimes$-module is a tensor product of evaluation modules. We give three characterizations that enable us to recognize the evaluation modules among all finite-dimensional irreducible $\boxtimes$-modules. Upon twisting an evaluation module via an element of S_4 we get an evaluation module whose evaluation parameter is potentially different. We describe how twisting affects the evaluation parameter. To do this, we first display an action of S_4 on $\mathbb{F}\backslash\{0,1\}$ by linear fractional transformations. We then show that for an evaluation module $V_d(a)$ and $\sigma \in S_4$, the $\boxtimes$-module $V_d(a)$ twisted via σ is isomorphic to $V_d(\sigma(a))$. Let G denote the kernel of the above S_4-action on $\mathbb{F}\backslash\{0,1\}$. We show that G is the unique normal subgroup of S_4 that has cardinality 4. For a given evaluation module $V_d(a)$ we consider 24 bases that are described as follows. Let $\mathbb{I}$ denote the vertex set of the tetrahedron. For mutually distinct $i,j,k,\ell \in \mathbb{I}$ we define an (ordered) basis $[i,j,k,\ell]$ of $V_d(a)$ such that (i) the basis diagonalizes the $\boxtimes$-generator identified with the edge of the tetrahedron incident with k and ℓ; (ii) the sum of the basis vectors is a common eigenvector for the three $\boxtimes$-generators identified with the edges incident with i. We find the matrices that represent all the $\boxtimes$-generators with respect to $[i,j,k,\ell]$. Moreover we find the transition matrices from the basis $[i,j,k,\ell]$ to the bases

$$[j,i,k,\ell], \qquad [i,k,j,\ell], \qquad [i,j,\ell,k].$$

The first matrix is diagonal, the second matrix is lower triangular, and the third matrix is the identity reflected about a vertical axis. We wind up our discussion of evaluation modules by showing how they can be concretely realized using homogeneous polynomials in two variables. Now consider a general finite-dimensional irreducible $\boxtimes$-module V. For $\sigma \in G$ we show that V twisted via σ is isomorphic to V. We associate with V two polynomials whose algebraic structure reflects that of V. The first polynomial, denoted S_V and called the shape polynomial, has coefficients that encode the eigenspace dimensions for the action of the $\boxtimes$-generators on V. We show how a certain factorization of S_V over the integers corresponds to the factorization of V into a tensor product of evaluation modules. The second polynomial, denoted P_V and called the Drinfel'd polynomial, is analogous to the Drinfel'd polynomial for the classical $\mathfrak{sl}_2$ loop algebra [4], [5], [23]. We show that the factorization of P_V into linear factors corresponds to the factorization of V into a tensor product of evaluation modules. Moreover we show that the map $V \mapsto P_V$ induces a bijection between (i) the set of isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules; (ii) the set of univariate polynomials over $\mathbb{F}$ that have constant coefficient 1 and do not vanish at 1. We display a nondegenerate bilinear form $\langle\, ,\, \rangle$ on V such that

$$\langle \xi.u, v\rangle = -\langle u, \xi.v\rangle \qquad \xi \in \boxtimes, \qquad u, v \in V.$$

We use this form to show that the $\boxtimes$-module V is isomorphic to the dual $\boxtimes$-module V^*. There is another type of bilinear form on V that is of interest. For a nonidentity $\sigma \in G$ we display a nondegenerate symmetric bilinear form $\langle\, ,\, \rangle_\sigma$ on V such that

$$\langle \xi.u, v\rangle_\sigma = -\langle u, \sigma(\xi).v\rangle_\sigma \qquad \xi \in \boxtimes, \qquad u, v \in V.$$

We remark on our motivation to investigate $\boxtimes$. This has to do with the tridiagonal pairs [18], [19] and the closely related Leonard pairs [29], [31], [32]. A Leonard pair is a pair of semisimple linear transformations on a finite-dimensional vector space, each of which acts in an irreducible tridiagonal fashion on an eigenbasis for the other [29, Definition 1.1]. There is a close connection between the Leonard pairs and the orthogonal polynomials that make up the terminating branch of the Askey scheme [22], [29], [30]. A tridiagonal pair is a mild generalization of a Leonard pair [18, Definition 1.1]. A pair of $\boxtimes$-generators identified with opposite edges of the tetrahedron act on each finite-dimensional irreducible $\boxtimes$-module as a tridiagonal pair [14, Theorem 1.7, Corollary 2.7]. For the details on this connection see [14, Section 2].

2 The three-point $\mathfrak{sl}_2$ loop algebra

Throughout the paper $\mathbb{F}$ denotes an algebraically closed field with characteristic 0. All unadorned tensor products are meant to be over $\mathbb{F}$. Recall that $\mathfrak{sl}_2$ is the Lie algebra over $\mathbb{F}$ with a basis e, f, h such that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Following [15, Lemma 3.2] we define

$$x = 2e - h, \qquad y = -2f - h, \qquad z = h. \tag{1}$$

Then x, y, z is a basis for $\mathfrak{sl}_2$ and

$$[x, y] = 2x + 2y, \qquad [y, z] = 2y + 2z, \qquad [z, x] = 2z + 2x.$$

By analogy with [21], [33] we call x, y, z the *equitable basis* for $\mathfrak{sl}_2$. See [2] for a detailed study of this basis.

Let $\mathcal{O}$ denote the Lie algebra over $\mathbb{F}$ defined by generators X, Y and relations

$$[X,[X,[X,Y]]] = 4[X,Y], \qquad [Y,[Y,[Y,X]]] = 4[Y,X]. \tag{2}$$

The algebra $\mathcal{O}$ is called the *Onsager algebra* [24], [26]. The equations (2) are known as the *Dolan-Grady* relations [8], [10], [11].

Definition 2.1 Let t denote an indeterminate and let $\mathbb{F}[t, t^{-1}, (t-1)^{-1}]$ denote the $\mathbb{F}$-algebra of all Laurent polynomials in $t, t-1$ that have coefficients in $\mathbb{F}$. We abbreviate

$$\mathcal{A} = \mathbb{F}[t, t^{-1}, (t-1)^{-1}].$$

We consider the Lie algebra over $\mathbb{F}$ consisting of the $\mathbb{F}$-vector space $\mathfrak{sl}_2 \otimes \mathcal{A}$ and Lie bracket

$$[u \otimes a, v \otimes b] = [u, v] \otimes ab, \qquad u, v \in \mathfrak{sl}_2, \qquad a, b \in \mathcal{A}.$$

We call $\mathfrak{sl}_2 \otimes \mathcal{A}$ the *three-point* $\mathfrak{sl}_2$ *loop algebra.*

See [3], [13], [27], [28] for information on multipoint loop algebras and related topics.

Definition 2.2 [15, Definition 1.1] Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \qquad \mathbb{I} = \{0, 1, 2, 3\} \tag{3}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{ij}, x_{jk}] = 2x_{ij} + 2x_{jk}.$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[x_{ij}, [x_{ij}, [x_{ij}, x_{k\ell}]]] = 4[x_{ij}, x_{k\ell}].$$

We call $\boxtimes$ the *tetrahedron algebra* or "tet" for short.

Proposition 2.3 [15, Proposition 6.5] *There exists an isomorphism of Lie algebras $\psi : \boxtimes \to \mathfrak{sl}_2 \otimes \mathcal{A}$ that sends*

$$\begin{aligned}
x_{12} &\mapsto x \otimes 1, & x_{03} &\mapsto y \otimes t + z \otimes (t-1), \\
x_{23} &\mapsto y \otimes 1, & x_{01} &\mapsto z \otimes (1 - t^{-1}) - x \otimes t^{-1}, \\
x_{31} &\mapsto z \otimes 1, & x_{02} &\mapsto x \otimes (1-t)^{-1} + y \otimes t(1-t)^{-1},
\end{aligned}$$

where x, y, z is the equitable basis for $\mathfrak{sl}_2$.

Lemma 2.4 [15, Corollary 12.1, Corollary 12.2] *For mutually distinct $i, j, k \in \mathbb{I}$ there exists an injective homomorphism of Lie algebras $\mathfrak{sl}_2 \to \boxtimes$ that sends*

$$x \mapsto x_{ij}, \qquad y \mapsto x_{jk}, \qquad z \mapsto x_{ki}.$$

For mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists an injective homomorphism of Lie algebras $\mathcal{O} \to \boxtimes$ that sends $X \mapsto x_{ij}$ and $Y \mapsto x_{k\ell}$.

3 Background on finite-dimensional $\boxtimes$-modules

In this section we review some basic facts and notation concerning finite-dimensional irreducible $\boxtimes$-modules. This material is summarized from [14].

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\{s_n\}_{n=0}^d$ denote a sequence of positive integers whose sum is the dimension of V. By a *decomposition of V of shape* $\{s_n\}_{n=0}^d$ we mean a sequence $\{V_n\}_{n=0}^d$ of subspaces of V such that V_n has dimension s_n for $0 \leq n \leq d$ and $V = \sum_{n=0}^d V_n$ (direct sum). We call V_n the nth *component* of the decomposition. We call d the *diameter* of the decomposition. For notational convenience we define $V_{-1} = 0$ and $V_{d+1} = 0$. By the *inversion* of $\{V_n\}_{n=0}^d$ we mean the decomposition $\{V_{d-n}\}_{n=0}^d$.

Now let V denote a finite-dimensional irreducible $\boxtimes$-module. By [14, Theorem 3.8] each generator x_{ij} of $\boxtimes$ is semisimple on V. Moreover there exists an integer $d \geq 0$ such that for each generator x_{ij} the set of distinct eigenvalues on V is $\{d - 2n \,|\, 0 \leq n \leq d\}$. We call d the *diameter* of V. For distinct $i, j \in \mathbb{I}$ we define a decomposition of V called $[i, j]$. The decomposition $[i, j]$ has diameter d. For $0 \leq n \leq d$ the nth component of $[i, j]$ is the eigenspace of x_{ij} for the eigenvalue $2n - d$. Using Definition 2.2(i) we find $[j, i]$ is the inversion of $[i, j]$. By [14, Corollary 3.6], for distinct $i, j \in \mathbb{I}$ the shape of $[i, j]$ is independent of i, j. We denote this shape by $\{\rho_n\}_{n=0}^d$ and note that $\rho_n = \rho_{d-n}$ for $0 \leq n \leq d$. By the *shape of V* we mean the sequence $\{\rho_n\}_{n=0}^d$.

For distinct $i, j \in \mathbb{I}$ and for distinct $r, s \in \mathbb{I}$ we now describe the action of x_{rs} on the decomposition $[i, j]$ of V. Denote this decomposition by $\{V_n\}_{n=0}^d$. Then by [14, Theorem 3.3], for $0 \leq n \leq d$ the action of x_{rs} on V_n is given in the table below.

Case		Action of x_{rs} on V_n
$r = i$,	$s = j$	$(x_{rs} - (2n - d)I)V_n = 0$
$r = j$,	$s = i$	$(x_{rs} - (d - 2n)I)V_n = 0$
$r = j$,	$s \neq i$	$(x_{rs} - (d - 2n)I)V_n \subseteq V_{n+1}$
$r \neq i$,	$s = j$	$(x_{rs} - (2n - d)I)V_n \subseteq V_{n+1}$
$r = i$,	$s \neq j$	$(x_{rs} - (2n - d)I)V_n \subseteq V_{n-1}$
$r \neq j$,	$s = i$	$(x_{rs} - (d - 2n)I)V_n \subseteq V_{n-1}$
i, j, r, s distinct		$x_{rs}V_n \subseteq V_{n-1} + V_n + V_{n+1}$

We recall the notion of a *flag*. For the moment let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let $\{s_n\}_{n=0}^d$ denote a sequence of positive integers whose sum is the dimension of V. By a *flag on V of shape* $\{s_n\}_{n=0}^d$ we mean a nested sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ of subspaces of V such that the dimension of U_n is $s_0 + \cdots + s_n$ for $0 \leq n \leq d$. We call U_n the nth *component* of the flag. We call d the *diameter* of the flag. We observe that $U_d = V$. The following construction yields a flag on V. Let $\{V_n\}_{n=0}^d$ denote a decomposition of V of shape $\{s_n\}_{n=0}^d$. Define

$$U_n = V_0 + V_1 + \cdots + V_n \qquad (0 \leq n \leq d).$$

Then the sequence $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ is a flag on V of shape $\{s_n\}_{n=0}^d$. We say this flag is *induced* by the decomposition $\{V_n\}_{n=0}^d$. Suppose we are given two flags on V with the same diameter: $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_d$ and $U'_0 \subseteq U'_1 \subseteq \cdots \subseteq U'_d$. We say these flags are *opposite* whenever there exists a decomposition $\{V_n\}_{n=0}^d$ of V such that

$$U_n = V_0 + V_1 + \cdots + V_n, \qquad\qquad U'_n = V_d + V_{d-1} + \cdots + V_{d-n}$$

for $0 \leq n \leq d$. In this case

$$U_i \cap U'_j = 0 \qquad \text{if} \quad i + j < d \qquad\qquad (0 \leq i, j \leq d) \tag{4}$$

and

$$V_n = U_n \cap U'_{d-n} \qquad\qquad (0 \leq n \leq d). \tag{5}$$

In particular the decomposition $\{V_n\}_{n=0}^d$ is uniquely determined by the given flags.

We now return our attention to $\boxtimes$-modules. Let V denote a finite-dimensional irreducible $\boxtimes$-module of diameter d. By [14, Lemma 5.3] there exists a collection of flags on V, denoted $[i], i \in \mathbb{I}$, such that for distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ induces $[i]$. By construction, for $i \in \mathbb{I}$ the the shape of the flag $[i]$ coincides with the shape of V. By [14, Lemma 5.6] the flags $[i]$, $i \in \mathbb{I}$ are mutually opposite. By [14, Lemma 5.7], for distinct $i, j \in \mathbb{I}$ and for $0 \leq n \leq d$ the nth component of $[i, j]$ is equal to the intersection of the following two sets:

(i) component n of the flag $[i]$;

(ii) component $d - n$ of the flag $[j]$.

4 Evaluation modules for $\boxtimes$

Lemma 4.1 *Let K denote an ideal of $\boxtimes$ such that $K \neq \boxtimes$. Then for mutually distinct $i, j, k \in \mathbb{I}$ the following sum is direct:*

$$\mathbb{F}x_{ij} + \mathbb{F}x_{jk} + \mathbb{F}x_{ki} + K. \tag{6}$$

Proof: We first claim that $x_{rs} \notin K$ for all distinct $r, s \in \mathbb{I}$. To prove the claim it suffices to show that $x_{rs} \in K$ implies $x_{su} \in K$ for mutually distinct $r, s, u \in \mathbb{I}$. If $x_{rs} \in K$ then $[x_{rs}, x_{su}] \in K$ since K is an ideal. Also $[x_{rs}, x_{su}] = 2x_{rs} + 2x_{su}$ by Definition 2.2(ii) so $x_{su} \in K$ and the claim follows. We can now easily show that the sum (6) is direct. By Lemma 2.4 the generators x_{ij}, x_{jk}, x_{ki} form a basis for a subalgebra L of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$. Recall that $\mathfrak{sl}_2$ is simple so L is simple. By this and since $K \cap L$ is an ideal of L we find $K \cap L = 0$ or $L \subseteq K$. The second possibility cannot occur by the claim, so $K \cap L = 0$ and this means that the sum (6) is direct. $\square$

Lemma 4.2 *For a nonzero Lie algebra homomorphism $\boxtimes \to \mathfrak{sl}_2$ and for mutually distinct $i, j, k \in \mathbb{I}$ the images of x_{ij}, x_{jk}, x_{ki} form a basis for $\mathfrak{sl}_2$.*

Proof: Let K denote the kernel of the homomorphism and note that K is an ideal of $\boxtimes$. The homomorphism is nonzero so $K \neq \boxtimes$. Applying Lemma 4.1 to K we find that the images of x_{ij}, x_{jk}, x_{ki} are linearly independent in $\mathfrak{sl}_2$. The result follows since $\mathfrak{sl}_2$ has dimension 3. $\square$

Proposition 4.3 *For a nonzero Lie algebra homomorphism $\boxtimes \to \mathfrak{sl}_2$ there exists a unique $a \in \mathbb{F}$ such that the following are in the kernel:*

$$ax_{01} + (1-a)x_{02} - x_{03}, \qquad ax_{10} + (1-a)x_{13} - x_{12},$$
$$ax_{23} + (1-a)x_{20} - x_{21}, \qquad ax_{32} + (1-a)x_{31} - x_{30}.$$

Moreover $a \neq 0$ and $a \neq 1$.

Proof: We first claim that for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the images of

$$x_{ij} - x_{ik}, \qquad x_{ik} - x_{i\ell}$$

in $\mathfrak{sl}_2$ are nonzero and linearly dependent. These images are nonzero by Definition 2.2(i) and Lemma 4.2. To show that they are linearly dependent, let K denote the kernel of the homomorphism. By Lemma 4.2 there exist scalars $\alpha, \beta, \gamma \in \mathbb{F}$ such that

$$x_{i\ell} - \alpha x_{ij} - \beta x_{jk} - \gamma x_{ki} \tag{7}$$

is in K. Taking the Lie bracket of (7) with x_{ij} and simplifying the result using Definition 2.2(i),(ii) we find

$$x_{i\ell} + (\beta - \gamma - 1)x_{ij} + \beta x_{jk} - \gamma x_{ki} \tag{8}$$

is in K. Comparing (7), (8) and using Lemma 4.2 we find $\beta = 0$ and $\gamma = \alpha - 1$. Evaluating (7) using this and Definition 2.2(i) we find that K contains

$$\alpha(x_{ij} - x_{ik}) + x_{ik} - x_{i\ell}$$

and the claim follows. We now apply the claim for the following values of (i, j, k, ℓ):

$$(0,1,2,3), \qquad (1,0,3,2), \qquad (2,3,0,1), \qquad (3,2,1,0).$$

We find that there exist unique scalars $a, b, c, d \in \mathbb{F}$ such that each of

$$ax_{01} + (1-a)x_{02} - x_{03}, \tag{9}$$
$$bx_{10} + (1-b)x_{13} - x_{12}, \tag{10}$$
$$cx_{23} + (1-c)x_{20} - x_{21}, \tag{11}$$
$$dx_{32} + (1-d)x_{31} - x_{30} \tag{12}$$

is in K. None of a, b, c, d is equal to 0 or 1 in view of Lemma 4.2 and Definition 2.2(i). Define $E \in \boxtimes$ to be (9) plus a/b times (10) plus $(1-a)/(1-c)$ times (11) plus (12). By construction $E \in K$. Also by Definition 2.2(i), E is equal to a certain linear combination of x_{12}, x_{23}, x_{31}. In this linear combination each coefficient must be 0 since the images of x_{12}, x_{23}, x_{31} are linearly independent in $\mathfrak{sl}_2$. Therefore

$$\frac{a}{b}, \qquad \frac{1-a}{1-c}, \qquad \frac{d}{c}, \qquad \frac{1-d}{1-b}$$

coincide and this yields $a = b = c = d$ after a brief calculation. The result follows. $\square$

We have a comment. Up to isomorphism there exists a unique $\boxtimes$-module with dimension 1, and every element of $\boxtimes$ is 0 on this $\boxtimes$-module. We call this $\boxtimes$-module *trivial*.

Lemma 4.4 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module. Then for mutually distinct $i, j, k \in \mathbb{I}$ the actions of x_{ij}, x_{jk}, x_{ki} on V are linearly independent.*

Proof: Let $K = \{\xi \in \boxtimes \,|\, \xi.v = 0 \;\forall v \in V\}$ be the kernel of the $\boxtimes$-action on V and note that K is an ideal of $\boxtimes$. By assumption V is nontrivial so $K \neq \boxtimes$. Applying Lemma 4.1 to K we obtain the result. $\square$

Lemma 4.5 *With reference to Definition 2.1 the following hold for $a \in \mathbb{F}\backslash\{0, 1\}$.*

(i) There exists an $\mathbb{F}$-algebra homomorphism $e_a : \mathcal{A} \to \mathbb{F}$ that sends $t \mapsto a$.

(ii) There exists a Lie algebra homomorphism $ev_a : \mathfrak{sl}_2 \otimes \mathcal{A} \to \mathfrak{sl}_2$ that sends $u \otimes b \to ue_a(b)$ for all $u \in \mathfrak{sl}_2$ and $b \in \mathcal{A}$. Moreover ev_a is surjective.

Proof: Routine. $\square$

Definition 4.6 For $a \in \mathbb{F}\backslash\{0, 1\}$ we define $EV_a : \boxtimes \to \mathfrak{sl}_2$ to be the composition $ev_a \circ \psi$, where ψ and ev_a are from Proposition 2.3 and Lemma 4.5 respectively. We note that EV_a is a surjective homomorphism of Lie algebras.

The map EV_a from Definition 4.6 is characterized as follows.

Lemma 4.7 *For a Lie algebra homomorphism $\varepsilon : \boxtimes \to \mathfrak{sl}_2$ the following are equivalent.*

(i) ε sends

$$x_{12} \mapsto x, \qquad x_{23} \mapsto y, \qquad x_{31} \mapsto z, \tag{13}$$

where x, y, z is the equitable basis for $\mathfrak{sl}_2$.

(ii) There exists $a \in \mathbb{F}\backslash\{0, 1\}$ such that $\varepsilon = EV_a$.

Suppose (i), (ii) hold. Then ε sends

$$x_{03} \mapsto ay + (a - 1)z, \tag{14}$$
$$x_{01} \mapsto (1 - a^{-1})z - a^{-1}x, \tag{15}$$
$$x_{02} \mapsto (1 - a)^{-1}x + a(1 - a)^{-1}y. \tag{16}$$

Proof: (i) $\Rightarrow$ (ii) The map ε is nonzero so Proposition 4.3 applies; let $a \in \mathbb{F}$ be from that proposition and note that $a \neq 0$, $a \neq 1$. We show that $\varepsilon = EV_a$. To this end we first show that ε satisfies (14)–(16). Line (14) follows from Definition 2.2(i), line (13), and the bottom-right display in Proposition 4.3. Line (15) follows from Definition 2.2(i), line (13), and the top-right display in Proposition 4.3. Line (16) follows from Definition 2.2(i), line (13), and the bottom-left display in Proposition 4.3. We have now shown that ε satisfies (14)–(16). Comparing the data in Proposition 2.3 with (13)–(16) we find $\varepsilon(x_{ij}) = EV_a(x_{ij})$ for all distinct $i, j \in \mathbb{I}$. It follows that $\varepsilon = EV_a$.
(ii) $\Rightarrow$ (i) Immediate from Proposition 2.3 and Definition 4.6.
Suppose (i), (ii) hold. We saw in the proof of (i) $\Rightarrow$ (ii) that ε satisfies (14)–(16). $\square$

Definition 4.8 For a finite-dimensional $\mathfrak{sl}_2$-module V and for $a \in \mathbb{F}\backslash\{0,1\}$ we pull back the $\mathfrak{sl}_2$-module action via EV_a to obtain a $\boxtimes$-module structure on V. We denote this $\boxtimes$-module by $V(a)$.

We now recall the finite-dimensional irreducible $\mathfrak{sl}_2$-modules.

Lemma 4.9 [16, p. 31] *For all integers $d \geq 0$, up to isomorphism there exists a unique irreducible $\mathfrak{sl}_2$-module V_d of dimension $d+1$. The module V_d has a basis $\{v_n\}_{n=0}^d$ such that*

$$\begin{aligned} e.v_n &= (d-n+1)v_{n-1} \qquad (1 \leq n \leq d), \quad e.v_0 = 0, \\ f.v_n &= (n+1)v_{n+1} \qquad (0 \leq n \leq d-1), \quad f.v_d = 0, \\ h.v_n &= (d-2n)v_n \qquad (0 \leq n \leq d). \end{aligned}$$

Lemma 4.10 *With respect to the basis $\{v_n\}_{n=0}^d$ for V_d given in Lemma 4.9, the elements x, y, z act as follows:*

$$\begin{aligned} (x+(d-2n)I).v_n &= 2(d-n+1)v_{n-1} \qquad (1 \leq n \leq d), \quad (x+dI).v_0 = 0, \\ (y+(d-2n)I).v_n &= -2(n+1)v_{n+1} \qquad (0 \leq n \leq d-1), \quad (y-dI).v_d = 0, \\ z.v_n &= (d-2n)v_n \qquad (0 \leq n \leq d). \end{aligned}$$

Proof: Use Lemma 4.9 and (1). □

Definition 4.11 With reference to Definition 4.8 and Lemma 4.9, by an *evaluation module* for $\boxtimes$ we mean a $\boxtimes$-module $V_d(a)$ where d is a positive integer and $a \in \mathbb{F}\backslash\{0,1\}$. We note that the $\boxtimes$-module $V_d(a)$ is nontrivial and irreducible, with diameter d and shape $(1,1,\ldots,1)$. We call a the *evaluation parameter* of $V_d(a)$.

Example 4.12 For $a \in \mathbb{F}\backslash\{0,1\}$ the $\boxtimes$-module $V_1(a)$ is described as follows. Let v_0, v_1 denote the basis for the $\mathfrak{sl}_2$-module V_1 given in Lemma 4.9. With respect to this basis and for the $\boxtimes$-module $V_1(a)$ the generators x_{ij} are represented by the following matrices.

$$\begin{aligned} x_{12} &: \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, & x_{03} &: \begin{pmatrix} -1 & 0 \\ -2a & 1 \end{pmatrix}, \\ x_{23} &: \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}, & x_{01} &: \begin{pmatrix} 1 & -2a^{-1} \\ 0 & -1 \end{pmatrix}, \\ x_{31} &: \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & x_{02} &: \begin{pmatrix} \frac{a+1}{a-1} & \frac{2}{1-a} \\ \frac{2a}{a-1} & \frac{1+a}{1-a} \end{pmatrix}. \end{aligned}$$

Proof: Combine Lemma 4.7 and Lemma 4.10. □

5 The evaluation modules for $\boxtimes$; three characterizations

Here is our first characterization of the evaluation modules for $\boxtimes$.

Proposition 5.1 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module. Then for $a \in \mathbb{F}\backslash\{0,1\}$ the following are equivalent.*

(i) V is isomorphic to an evaluation module with evaluation parameter a.

(ii) Each of the following vanishes on V:

$$ax_{01} + (1-a)x_{02} - x_{03}, \qquad ax_{10} + (1-a)x_{13} - x_{12}, \tag{17}$$

$$ax_{23} + (1-a)x_{20} - x_{21}, \qquad ax_{32} + (1-a)x_{31} - x_{30}. \tag{18}$$

Proof: (i) $\Rightarrow$ (ii) Immediate from Lemma 4.7.
(ii) $\Rightarrow$ (i) By Lemma 2.4 the generators x_{12}, x_{23}, x_{31} form a basis for a subalgebra L of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$. Let K denote the ideal of $\boxtimes$ generated by the four elements in (17), (18). We claim that K is equal to the kernel of the $\boxtimes$-action on V. To prove the claim, let K' denote the kernel in question and observe that $K \subseteq K'$. Note that K' is an ideal of $\boxtimes$; also $K' \neq \boxtimes$ since V is nontrivial. Applying Lemma 4.1 to K' we obtain $K' \cap L = 0$. Note that $K + L$ is a subalgebra of $\boxtimes$ that contains the generators x_{ij} $(i, j \in \mathbb{I},\ i \neq j)$. Therefore $K + L = \boxtimes$. By these comments $K = K'$ and the claim is proved. We now show that the $\boxtimes$-modules V and $V_d(a)$ are isomorphic, where $d+1$ is the dimension of V. For V and $V_d(a)$ we restrict the $\boxtimes$-action to L. We mentioned earlier that the $\boxtimes$-module $V_d(a)$ is irreducible. Now the L-module $V_d(a)$ is irreducible since $K + L = \boxtimes$ and K vanishes on $V_d(a)$. Similarly the L-module V is irreducible. Since L is isomorphic to $\mathfrak{sl}_2$ and since V, $V_d(a)$ have the same dimension the L-modules V, $V_d(a)$ are isomorphic. Let $\mu : V \to V_d(a)$ denote an isomorphism of L-modules. Since $K + L = \boxtimes$ and K vanishes on each of V, $V_d(a)$ the map μ extends to a $\boxtimes$-module isomomorphism from V to $V_d(a)$. Therefore the $\boxtimes$-modules V and $V_d(a)$ are isomorphic as desired. $\square$

Lemma 5.2 *Two evaluation modules for $\boxtimes$ are isomorphic if and only if they have the same diameter and the same evaluation parameter.*

Proof: Two isomorphic evaluation modules have the same diameter, since they have the same dimension and the dimension is one more than the diameter. Suppose we are given isomorphic evaluation modules $V_d(a)$ and $V_d(b)$. We show $a = b$. By (17) the element $ax_{01} + (1-a)x_{02} - x_{03}$ vanishes on $V_d(a)$. By (17) and since $V_d(a), V_d(b)$ are isomorphic the element $bx_{01} + (1-b)x_{02} - x_{03}$ vanishes on $V_d(a)$. Comparing these elements we find that $a = b$ or $x_{01} - x_{02}$ vanishes on $V_d(a)$. The second possibility contradicts Definition 2.2(i) and Lemma 4.4, so $a = b$ and the result follows. $\square$

Here is the second characterization of the evaluation modules for $\boxtimes$.

Proposition 5.3 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module. Then the following are equivalent.*

(i) V is isomorphic to an evaluation module for $\boxtimes$.

(ii) There exist mutually distinct $i, j, k, \ell \in \mathbb{I}$ such that $x_{ij}, x_{ik}, x_{i\ell}$ are linearly dependent on V.

Proof: (i) $\Rightarrow$ (ii) Immediate from Proposition 5.1.
(ii) $\Rightarrow$ (i) Without loss we assume $(i, j, k, \ell) = (0, 1, 2, 3)$. By assumption there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$, not all zero, such that

$$\alpha_1 x_{01} + \alpha_2 x_{02} + \alpha_3 x_{03} = 0 \tag{19}$$

on V. Each of $\alpha_1, \alpha_2, \alpha_3$ is nonzero by Definition 2.2(i) and Lemma 4.4. Taking the Lie bracket of x_{01} with (19) and simplifying the result using Definition 2.2(i),(ii) we find that

$$(\alpha_2 + \alpha_3)x_{01} - \alpha_2 x_{02} - \alpha_3 x_{03} = 0 \tag{20}$$

on V. Adding (19), (20) and using Lemma 4.4 we find

$$\alpha_1 + \alpha_2 + \alpha_3 = 0. \tag{21}$$

Taking the Lie bracket of x_{12} with (19) and simplifying the result using Definition 2.2(i),(ii) and (21) we find

$$[x_{12}, x_{03}] = 2\alpha_1\alpha_3^{-1}x_{01} - 2\alpha_2\alpha_3^{-1}x_{02} - 2x_{12} \tag{22}$$

on V. Cyclically permuting $1, 2, 3$ in the previous argument we find that both

$$[x_{23}, x_{01}] = 2\alpha_2\alpha_1^{-1}x_{02} - 2\alpha_3\alpha_1^{-1}x_{03} - 2x_{23}, \tag{23}$$
$$[x_{31}, x_{02}] = 2\alpha_3\alpha_2^{-1}x_{03} - 2\alpha_1\alpha_2^{-1}x_{01} - 2x_{31} \tag{24}$$

on V. By the Jacobi identity

$$[x_{01}, [x_{12}, x_{23}]] + [x_{12}, [x_{23}, x_{01}]] + [x_{23}, [x_{01}, x_{12}]] = 0. \tag{25}$$

Evaluating (25) using Definition 2.2(i),(ii) and (21)–(23) we find that

$$\alpha_1 x_{23} + \alpha_2 x_{20} + \alpha_3 x_{21} = 0 \tag{26}$$

on V. Cyclically permuting $1, 2, 3$ in the previous argument we find that both

$$\alpha_2 x_{31} + \alpha_3 x_{30} + \alpha_1 x_{32} = 0, \tag{27}$$
$$\alpha_3 x_{12} + \alpha_1 x_{10} + \alpha_2 x_{13} = 0 \tag{28}$$

on V. By (19), (21) and (26)–(28) the four expressions (17), (18) vanish on V, where $a = -\alpha_1\alpha_3^{-1}$. Note that $a \neq 0$ since $\alpha_1 \neq 0$, and $a \neq 1$ since $\alpha_2 \neq 0$. Now by Proposition 5.1, V is isomorphic to an evaluation module for $\boxtimes$ with evaluation parameter a. The result follows. $\square$

Here is the third characterization of the evaluation modules for $\boxtimes$.

Proposition 5.4 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module. Then the following are equivalent.*

(i) V is isomorphic to an evaluation module for $\boxtimes$.

(ii) V has shape $(1, 1, \ldots, 1)$.

Proof: (i) $\Rightarrow$ (ii) By Definition 4.11 each evaluation module for $\boxtimes$ has shape $(1, 1, \ldots, 1)$.
(ii) $\Rightarrow$ (i) Throughout this proof the decomposition $[1, 3]$ of V will be denoted by $\{V_n\}_{n=0}^d$. By definition, for $0 \le n \le d$ the space V_n is an eigenspace for x_{13} with eigenvalue $2n - d$. By this and since $x_{13} + x_{31} = 0$, the space V_n is an eigenspace for x_{31} with eigenvalue $d - 2n$. Also V_n has dimension 1 by our assumption on the shape. We will need the action of x_{30} on $\{V_n\}_{n=0}^d$. By the table in Section 3,

$$(x_{30} - (d - 2n)I)V_n \subseteq V_{n+1} \qquad (0 \le n \le d), \tag{29}$$

where we recall $V_{d+1} = 0$. By Lemma 2.4 there exists an injective homomorphism of Lie algebras from $\mathfrak{sl}_2$ to $\boxtimes$ that sends x, y, z to x_{12}, x_{23}, x_{31} respectively. Using this homomorphism we pull back the $\boxtimes$-module structure on V to obtain an $\mathfrak{sl}_2$-module structure on V. Since z and x_{31} agree on V we find that for $0 \le n \le d$, V_n is an eigenspace for z with eigenvalue $d - 2n$. These eigenspaces all have dimension 1 so the $\mathfrak{sl}_2$-module V is irreducible. Therefore the $\mathfrak{sl}_2$-module V is isomorphic to the $\mathfrak{sl}_2$-module V_d from Lemma 4.9. Let $\{v_n\}_{n=0}^d$ denote the basis for V given in Lemma 4.9. By Lemma 4.10 we find that for $0 \le n \le d$ the vector v_n is an eigenvector for z with eigenvalue $d - 2n$; therefore v_n is a basis for V_n. By Lemma 4.10 and since y, x_{23} agree on V, we find $(x_{23} + (d - 2n)I).v_n = -2(n + 1)v_{n+1}$ for $0 \le n \le d - 1$ and $(x_{23} - dI).v_d = 0$. By (29) there exist scalars $\{\alpha_n\}_{n=1}^d$ in $\mathbb{F}$ such that $(x_{30} - (d - 2n)I).v_n = \alpha_{n+1}v_{n+1}$ for $0 \le n \le d - 1$ and $(x_{30} + dI).v_d = 0$. By Definition 2.2(ii),

$$[x_{23}, x_{30}] = 2x_{23} + 2x_{30}.$$

For $0 \le n \le d$ we apply each side of this equation to v_n and evaluate the result using the above comments. The resulting equations show that α_n/n is independent of n for $1 \le n \le d$. Denoting this common value by $2a$ we find $(x_{30} - (d - 2n)I).v_n = 2a(n + 1)v_{n+1}$ for $0 \le n \le d - 1$. ¿From our comments so far and since $x_{23} + x_{32} = 0$, the element

$$ax_{32} + (1 - a)x_{31} - x_{30}$$

vanishes on each of $\{v_n\}_{n=0}^d$ and hence on V. In particular the actions of x_{30}, x_{31}, x_{32} on V are linearly dependent. Invoking Proposition 5.3 we find V is isomorphic to an evaluation module for $\boxtimes$. $\square$

6 Some automorphisms of $\boxtimes$

In this section we consider how the symmetric group S_4 acts on $\boxtimes$ as a group of automorphisms. We investigate what happens when we twist an evaluation module for $\boxtimes$ via an element of S_4.

For the rest of this paper we identify S_4 with the group of permutations of $\mathbb{I}$. We use the cycle notation; for example $(1, 2, 3)$ denotes the element of S_4 that sends $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $0 \mapsto 0$. Note that S_4 acts on the set of generators for $\boxtimes$ by permuting the indices:

$$\sigma(x_{ij}) = x_{\sigma(i),\sigma(j)} \qquad \sigma \in S_4, \qquad i, j \in \mathbb{I}, \quad i \neq j.$$

This action leaves invariant the defining relations for $\boxtimes$ and therefore induces an action of S_4 on $\boxtimes$ as a group of automorphisms. This S_4-action on $\boxtimes$ induces an action of S_4 on the set of $\boxtimes$-modules, as we now explain.

Definition 6.1 Let V denote a $\boxtimes$-module. For $\sigma \in S_4$ there exists a $\boxtimes$-module structure on V, called V *twisted via* σ, that behaves as follows: for all $\xi \in \boxtimes$ and $v \in V$, the vector $\xi.v$ computed in V twisted via σ coincides with the vector $\sigma^{-1}(\xi).v$ computed in the original $\boxtimes$-module V. Sometimes we abbreviate ${}^{\sigma}V$ for V twisted via σ. Observe that S_4 acts on the set of $\boxtimes$-modules, with σ sending V to ${}^{\sigma}V$ for all $\sigma \in S_4$ and all $\boxtimes$-modules V.

For the following three lemmas the proofs are routine and left to the reader.

Lemma 6.2 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For distinct $i, j \in \mathbb{I}$ and for $\sigma \in S_4$ the following coincide:*

(i) the decomposition $[i, j]$ of V;

(ii) the decomposition $[\sigma(i), \sigma(j)]$ of V twisted via σ.

Lemma 6.3 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For $i \in \mathbb{I}$ and for $\sigma \in S_4$ the following coincide:*

(i) the flag $[i]$ of V;

(ii) the flag $[\sigma(i)]$ of V twisted via σ.

Lemma 6.4 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For $\sigma \in S_4$ the following coincide:*

(i) the shape of V;

(ii) the shape of V twisted via σ.

Corollary 6.5 *Let V denote an evaluation module for $\boxtimes$, and pick $\sigma \in S_4$. Then V twisted via σ is isomorphic to an evaluation module for $\boxtimes$.*

Proof: Combine Proposition 5.4 and Lemma 6.4. □

For each integer $d \geq 1$ let Δ_d denote the set of isomorphism classes of evaluation modules for $\boxtimes$ that have diameter d. By Definition 4.11 and Lemma 5.2 the map $a \mapsto V_d(a)$ induces a bijection from $\mathbb{F}\backslash\{0, 1\}$ to Δ_d. Via this bijection the S_4-action on Δ_d from Corollary 6.5 induces an S_4-action on $\mathbb{F}\backslash\{0, 1\}$. This action is described in the following lemma and subsequent theorem.

Lemma 6.6 *There exists an S_4-action on the set $\mathbb{F}\backslash\{0,1\}$ that does the following. For all $a \in \mathbb{F}\backslash\{0,1\}$,*

(i) $(2,0)$ *sends* $a \mapsto a^{-1}$*;*

(ii) $(0,1)$ *sends* $a \mapsto a(a-1)^{-1}$*;*

(iii) $(1,3)$ *sends* $a \mapsto a^{-1}$*.*

Proof: Consider the following maps on $\mathbb{F}\backslash\{0,1\}$:

$$\sigma_1 : \ a \mapsto a^{-1}, \qquad \sigma_2 : \ a \mapsto a(a-1)^{-1}, \qquad \sigma_3 : \ a \mapsto a^{-1}.$$

One routinely verifies that $\sigma_1^2 = 1$, $\sigma_2^2 = 1$, $\sigma_3^2 = 1$ and that $(\sigma_1\sigma_2)^3 = 1$, $(\sigma_2\sigma_3)^3 = 1$, $(\sigma_1\sigma_3)^2 = 1$. The above equations are the defining relations for a well-known presentation for S_4 [17, p. 105] and the result follows. □

Theorem 6.7 *For an integer $d \geq 1$, for $\sigma \in S_4$, and for $a \in \mathbb{F}\backslash\{0,1\}$ the following are isomorphic:*

(i) the $\boxtimes$-module $V_d(a)$ twisted via σ;

(ii) the $\boxtimes$-module $V_d(\sigma(a))$.

Proof: We abbreviate $W = V_d(a)$. Without loss we assume that σ is one of $(2,0)$, $(0,1)$, $(1,3)$. In each case we verify using Proposition 5.1 that each of the following vanishes on W:

$$\begin{aligned}
&\sigma(a)x_{\sigma(0),\sigma(1)} + (1-\sigma(a))x_{\sigma(0),\sigma(2)} - x_{\sigma(0),\sigma(3)},\\
&\sigma(a)x_{\sigma(1),\sigma(0)} + (1-\sigma(a))x_{\sigma(1),\sigma(3)} - x_{\sigma(1),\sigma(2)},\\
&\sigma(a)x_{\sigma(2),\sigma(3)} + (1-\sigma(a))x_{\sigma(2),\sigma(0)} - x_{\sigma(2),\sigma(1)},\\
&\sigma(a)x_{\sigma(3),\sigma(2)} + (1-\sigma(a))x_{\sigma(3),\sigma(1)} - x_{\sigma(3),\sigma(0)}.
\end{aligned}$$

Now by Definition 6.1 and $\sigma^2 = 1$, each of the following vanishes on ${}^{\sigma}W$:

$$\begin{aligned}
&\sigma(a)x_{01} + (1-\sigma(a))x_{02} - x_{03}, \qquad && \sigma(a)x_{10} + (1-\sigma(a))x_{13} - x_{12},\\
&\sigma(a)x_{23} + (1-\sigma(a))x_{20} - x_{21}, \qquad && \sigma(a)x_{32} + (1-\sigma(a))x_{31} - x_{30}.
\end{aligned}$$

Now by Proposition 5.1 the $\boxtimes$-module ${}^{\sigma}W$ has evaluation parameter $\sigma(a)$ and is therefore isomorphic to $V_d(\sigma(a))$. □

In Lemma 6.6 we gave an action of S_4 on $\mathbb{F}\backslash\{0,1\}$. We now find the kernel of this action.

Definition 6.8 A partition of $\mathbb{I}$ into two disjoint sets, each with two elements, is said to have shape $(2,2)$. Let P denote the set of partitions of $\mathbb{I}$ that have shape $(2,2)$, and note that P has cardinality 3. The action of S_4 on $\mathbb{I}$ induces an action of S_4 on P and this yields a surjective homomorphism of groups $S_4 \to S_3$. Let G denote the kernel of this homomorphism. We note that G has cardinality 4 and consists of

$$(01)(23), \qquad (02)(13), \qquad (03)(12)$$

together with the identity element. Observe that G is isomorphic to the Klein 4-group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 6.9 *With reference to Definition 6.6, for $\sigma \in S_4$ the following are equivalent:*

(i) $\sigma \in G$;

(ii) $\sigma(a) = a$ *for all* $a \in \mathbb{F}\backslash\{0, 1\}$.

Proof: Let G' denote the set of elements in S_4 that fix each element of $\mathbb{F}\backslash\{0, 1\}$. We show $G' = G$. The set $\{3, 1/3, -2, -1/2, 3/2, 2/3\}$ is an orbit of S_4; therefore the index $[S_4 : G'] \geq 6$ so G' has at most 4 elements. Also G' contains $(2, 0)(1, 3)$ by Lemma 6.6 and G' is normal in S_4 so $G \subseteq G'$. It follows that $G' = G$. □

Corollary 6.10 *For an integer $d \geq 1$, for $\sigma \in G$, and for $a \in \mathbb{F}\backslash\{0, 1\}$ the following are isomorphic:*

(i) the $\boxtimes$-module $V_d(a)$ twisted via σ;

(ii) the $\boxtimes$-module $V_d(a)$.

Proof: Combine Theorem 6.7 and Lemma 6.9. □

In Lemma 6.6 we considered an S_4-action on the set $\mathbb{F}\backslash\{0, 1\}$. We now describe the orbits of this action.

Definition 6.11 Pick $a \in \mathbb{F}\backslash\{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$. By the (i, j, k, ℓ)-*relative of* a we mean the scalar $\sigma(a)$ where $\sigma \in S_4$ satisfies $\sigma(i) = 2$, $\sigma(j) = 0$, $\sigma(k) = 1$, $\sigma(\ell) = 3$.

Lemma 6.12 *Pick $a \in \mathbb{F}\backslash\{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$. Let α denote the (i, j, k, ℓ)-relative of a. Then the following hold.*

(i) α^{-1} is the (j, i, k, ℓ)-relative of a;

(ii) $\alpha(\alpha - 1)^{-1}$ is the (i, k, j, ℓ)-relative of a;

(iii) α^{-1} is the (i, j, ℓ, k)-relative of a.

Proof: By Definition 6.11 there exists $\sigma \in S_4$ that sends the sequence $(a; i, j, k, \ell)$ to $(\alpha; 2, 0, 1, 3)$. To obtain (i), note that $(2, 0)\sigma$ sends $(a; j, i, k, \ell)$ to $(\alpha^{-1}; 2, 0, 1, 3)$. To obtain (ii), note that $(0, 1)\sigma$ sends $(a; i, k, j, \ell)$ to $(\alpha(\alpha - 1)^{-1}; 2, 0, 1, 3)$. To obtain (iii), note that $(1, 3)\sigma$ sends $(a; i, j, \ell, k)$ to $(\alpha^{-1}; 2, 0, 1, 3)$. □

In the following lemma we interpret each relative as a cross ratio (see [6, p. 48] for the details).

Lemma 6.13 *For $a \in \mathbb{F}\backslash\{0, 1\}$ and mutually distinct $i, j, k, \ell \in \mathbb{I}$ the following coincide:*

(i) the (i, j, k, ℓ)-relative of a;

(ii) the scalar $\frac{\hat{i}-\hat{\ell}}{\hat{i}-\hat{k}}\frac{\hat{j}-\hat{k}}{\hat{j}-\hat{\ell}}$ *where we define* $\hat{0}=a$, $\hat{1}=0$, $\hat{2}=1$, $\hat{3}=\infty$.

Proof: Let us call the scalar in (ii) the (i,j,k,ℓ)-*partner* of a. Denoting this partner by α we observe:

- α^{-1} is the (j,i,k,ℓ)-partner of a;
- $\alpha(\alpha-1)^{-1}$ is the (i,k,j,ℓ)-partner of a;
- α^{-1} is the (i,j,ℓ,k)-partner of a.

Note that a is the $(2,0,1,3)$-partner of a as well as the $(2,0,1,3)$-relative of a. By these comments and Lemma 6.12 we find that the partner function and the relative function satisfy the same recursion and same initial condition. Therefore these functions coincide and the result follows. $\square$

Proposition 6.14 *Pick* $a\in\mathbb{F}\backslash\{0,1\}$ *and mutually distinct* $i,j,k,\ell\in\mathbb{I}$. *Then the* (i,j,k,ℓ)-*relative of* a *is given in the following table.*

(i,j,k,ℓ)				(i,j,k,ℓ)-relative
$(2,0,1,3)$	$(0,2,3,1)$	$(1,3,2,0)$	$(3,1,0,2)$	a
$(0,2,1,3)$	$(2,0,3,1)$	$(1,3,0,2)$	$(3,1,2,0)$	a^{-1}
$(1,0,2,3)$	$(0,1,3,2)$	$(2,3,1,0)$	$(3,2,0,1)$	$1-a$
$(0,1,2,3)$	$(1,0,3,2)$	$(2,3,0,1)$	$(3,2,1,0)$	$(1-a)^{-1}$
$(2,1,0,3)$	$(1,2,3,0)$	$(0,3,2,1)$	$(3,0,1,2)$	$a(a-1)^{-1}$
$(1,2,0,3)$	$(2,1,3,0)$	$(0,3,1,2)$	$(3,0,2,1)$	$1-a^{-1}$

Proof: Routine calculation using Lemma 6.12 or Lemma 6.13. $\square$

7 24 bases for an evaluation module

Let V denote an evaluation module for $\boxtimes$. In this section we display 24 bases for V that we find attractive. We compute the action of the generators x_{ij} on each of these bases. We find the transition matrices between certain pairs of bases among the 24.

Definition 7.1 Let V denote an evaluation module for $\boxtimes$ and pick mutually distinct $i,j,k,\ell\in\mathbb{I}$. A basis $\{u_n\}_{n=0}^d$ of V is called an $[i,j,k,\ell]$-*basis* whenever:

(i) for $0\le n\le d$ the vector u_n is contained in component n of the decomposition $[k,\ell]$ for V;

(ii) $\sum_{n=0}^d u_n$ is contained in component 0 of the flag $[i]$ for V.

Referring to Definition 7.1, we will show that V has an $[i, j, k, \ell]$-basis after a few comments.

Lemma 7.2 *Let V denote an evaluation module for $\boxtimes$ and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. Let $\{u_n\}_{n=0}^d$ denote an $[i, j, k, \ell]$-basis for V, and for $0 \le n \le d$ let u'_n denote any vector in V. Then the following are equivalent:*

(i) the sequence $\{u'_n\}_{n=0}^d$ is an $[i, j, k, \ell]$-basis for V;

(ii) there exists a nonzero $\beta \in \mathbb{F}$ such that $u'_n = \beta u_n$ for $0 \le n \le d$.

Proof: (i) $\Rightarrow$ (ii) Recall that V has shape $(1, 1, \ldots, 1)$.
(ii) $\Rightarrow$ (i) Immediate from Definition 7.1. $\square$

Lemma 7.3 Let $V_d(a)$ denote an evaluation module for $\boxtimes$. Then the basis $\{v_n\}_{n=0}^d$ for the $\mathfrak{sl}_2$-module V_d from Definition 4.9 is a $[2, 0, 1, 3]$-basis for $V_d(a)$.

Proof: By (13) and since $x_{13} = -x_{31}$ we find $z + x_{13}$ vanishes on $V_d(a)$. Now by Lemma 4.10, for $0 \le n \le d$ we have $x_{13}.v_n = (2n - d)v_n$ so v_n is contained in component n of the decomposition $[1, 3]$ for $V_d(a)$. Define $\eta = \sum_{n=0}^d v_n$. Using (13) and Lemma 4.10 we find $x_{23}.\eta = -d\eta$; therefore η is contained in component 0 of the flag $[2]$ for $V_d(a)$. Now the basis $\{v_n\}_{n=0}^d$ is a $[2, 0, 1, 3]$-basis by Definition 7.1. $\square$

Lemma 7.4 *Let V denote an evaluation module for $\boxtimes$ and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. Then for $\sigma \in S_4$ the following are the same:*

(i) an $[i, j, k, \ell]$-basis for V;

(ii) a $[\sigma(i), \sigma(j), \sigma(k), \sigma(\ell)]$-basis for V twisted via σ.

Proof: Use Lemma 6.2 and Lemma 6.3. $\square$

Proposition 7.5 *Let V denote an evaluation module for $\boxtimes$ and pick mutually distinct $i, j, k, \ell \in \mathbb{I}$. Then there exists an $[i, j, k, \ell]$-basis for V.*

Proof: Let σ denote the element of S_4 that sends the sequence (i, j, k, ℓ) to $(2, 0, 1, 3)$. The $\boxtimes$-module ${}^{\sigma}V$ is isomorphic to an evaluation module by Corollary 6.5, so ${}^{\sigma}V$ has a $[2, 0, 1, 3]$-basis by Lemma 7.3. Now V has an $[i, j, k, \ell]$-basis in view of Lemma 7.4. $\square$

We now find the action of each generator x_{rs} on the bases given in Proposition 7.5. To describe these actions we use the following notation. For an integer $d \ge 0$ let $\mathrm{Mat}_{d+1}(\mathbb{F})$ denote the $\mathbb{F}$-algebra of all $d + 1$ by $d + 1$ matrices that have entries in $\mathbb{F}$. We index the rows and columns by $0, 1, \ldots, d$. Let V denote a vector space over $\mathbb{F}$ with dimension $d + 1$ and let $\{u_n\}_{n=0}^d$ denote a basis for V. For a linear transformation $A : V \to V$ there exists a unique $M \in \mathrm{Mat}_{d+1}(\mathbb{F})$ such that $A.u_j = \sum_{i=0}^d M_{ij} u_i$ for $0 \le j \le d$. We call M the *matrix that represents A with respect to $\{u_n\}_{n=0}^d$*.

Lemma 7.6 *Let $V_d(a)$ denote an evaluation module for $\boxtimes$. For mutually distinct $i, j, k, \ell \in \mathbb{I}$, for distinct $r, s \in \mathbb{I}$, and for $\sigma \in S_4$ the following are the same:*

(i) the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$-basis for $V_d(a)$;

(ii) the matrix that represents $x_{\sigma(r),\sigma(s)}$ with respect to a $[\sigma(i), \sigma(j), \sigma(k), \sigma(\ell)]$-basis for $V_d(\sigma(a))$.

Proof: Routine using Definition 6.1, Theorem 6.7, and Lemma 7.4. □

Theorem 7.7 *Let $V_d(a)$ denote an evaluation module for $\boxtimes$. For mutually distinct $i, j, k, \ell \in \mathbb{I}$ and distinct $r, s \in \mathbb{I}$ consider the matrix that represents x_{rs} with respect to an $[i, j, k, \ell]$-basis for $V_d(a)$. The entries of this matrix are given in the following table. All entries not displayed are zero.*

generator	$(n, n-1)$-entry	(n, n)-entry	$(n-1, n)$-entry
$x_{\ell k}$	0	$d - 2n$	0
$x_{k\ell}$	0	$2n - d$	0
x_{ki}	0	$2n - d$	$2d - 2n + 2$
x_{ik}	0	$d - 2n$	$2n - 2d - 2$
$x_{i\ell}$	$-2n$	$2n - d$	0
$x_{\ell i}$	$2n$	$d - 2n$	0
$x_{\ell j}$	$2\alpha n$	$d - 2n$	0
$x_{j\ell}$	$-2\alpha n$	$2n - d$	0
x_{jk}	0	$d - 2n$	$2(n - d - 1)\alpha^{-1}$
x_{kj}	0	$2n - d$	$2(d - n + 1)\alpha^{-1}$
x_{ji}	$2\alpha n(\alpha - 1)^{-1}$	$(d - 2n)(\alpha + 1)(\alpha - 1)^{-1}$	$2(d - n + 1)(1 - \alpha)^{-1}$
x_{ij}	$2\alpha n(1 - \alpha)^{-1}$	$(d - 2n)(\alpha + 1)(1 - \alpha)^{-1}$	$2(d - n + 1)(\alpha - 1)^{-1}$

In the above table the scalar α denotes the (i, j, k, ℓ)-relative of a from Definition 6.11.

Proof: First consider the special case $(i, j, k, \ell) = (2, 0, 1, 3)$. Without loss we take our $[2, 0, 1, 3]$-basis to be the basis $\{v_n\}_{n=0}^d$ from Lemma 7.3. We then obtain the action of x_{rs} on $\{v_n\}_{n=0}^d$ using Lemma 4.7 and Lemma 4.10. This gives the result for the special case. To get the result for general (i, j, k, ℓ), let σ denote the element of S_4 that sends the sequence $(a; i, j, k, \ell)$ to $(\alpha; 2, 0, 1, 3)$. Using σ and Lemma 7.6 we reduce to the above special case. □

Let V denote an evaluation module for $\boxtimes$ and consider the 24 bases for V from Definition 7.1. Shortly we will display the transition matrices between certain pairs of bases among these 24. In order to do this, it is convenient to introduce a bilinear form on V.

We recall a few concepts from linear algebra. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *bilinear form* on V we mean a map $\langle \, , \, \rangle : V \times V \to \mathbb{F}$ that satisfies the following four conditions for all $u, v, w \in V$ and for all $\alpha \in \mathbb{F}$: (i) $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$; (ii) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$; (iii) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$; (iv) $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$. We observe that a scalar multiple of a bilinear form on V is a bilinear form on V. Let $\langle\, ,\, \rangle$ denote a bilinear form on V. This form is said to be *symmetric* (resp. *antisymmetric*) whenever $\langle u, v\rangle = \langle v, u\rangle$ (resp. $\langle u, v\rangle = -\langle v, u\rangle$) for all $u, v \in V$. Let $\langle\, ,\, \rangle$ denote a bilinear form on V. Then the following are equivalent: (i) there exists a nonzero $u \in V$ such that $\langle u, v\rangle = 0$ for all $v \in V$; (ii) there exists a nonzero $v \in V$ such that $\langle u, v\rangle = 0$ for all $u \in V$. The form $\langle\, ,\, \rangle$ is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. Assume $\langle\, ,\, \rangle$ is nondegenerate. Let $\{U_n\}_{n=0}^d$ and $\{V_n\}_{n=0}^d$ denote decompositions of V that have the same diameter. These decompositions are called *dual* with respect to $\langle\, ,\, \rangle$ whenever $\langle U_i, V_j\rangle = 0$ for $0 \le i, j \le d, i \ne j$. In this case $\{U_n\}_{n=0}^d$ and $\{V_n\}_{n=0}^d$ have the same shape.

Lemma 7.8 *For each integer $d \ge 0$ there exists a nonzero bilinear form $\langle\, ,\, \rangle$ on the $\mathfrak{sl}_2$-module V_d such that*

$$\langle \xi.u, v\rangle = -\langle u, \xi.v\rangle \qquad\qquad \xi \in \mathfrak{sl}_2, \qquad u, v \in V_d.$$

The form is nondegenerate. The form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. The form is symmetric (resp. antisymmetric) when d is even (resp. odd).

Proof: Concerning existence, let $\{v_n\}_{n=0}^d$ denote the basis for V_d from Lemma 4.9, and let $\langle\, ,\, \rangle$ denote the bilinear form on V_d that satisfies

$$\langle v_r, v_s\rangle = \delta_{r+s,d}(-1)^r \binom{d}{r} \qquad\qquad (0 \le r, s \le d).$$

Using the data in Lemma 4.9 we find $\langle \xi.v_r, v_s\rangle = -\langle v_r, \xi.v_s\rangle$ for $\xi \in \{e, f, h\}$ and $0 \le r, s \le d$. Since $\{v_n\}_{n=0}^d$ is a basis for V_d and e, f, h is a basis for $\mathfrak{sl}_2$ we conclude $\langle \xi.u, v\rangle = -\langle u, \xi.v\rangle$ for $\xi \in \mathfrak{sl}_2$ and $u, v \in V_d$. This shows that the required bilinear form exists. The remaining assertions are routinely verified. $\square$

Lemma 7.9 *Let V denote an evaluation module for $\boxtimes$. Then there exists a nonzero bilinear form $\langle\, ,\, \rangle$ on V such that*

$$\langle \xi.u, v\rangle = -\langle u, \xi.v\rangle \qquad\qquad \xi \in \boxtimes, \qquad u, v \in V.$$

The form is nondegenerate. The form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. The form is symmetric (resp. antisymmetric) when the diameter of V is even (resp. odd).

Proof: By definition there exist an integer $d \ge 1$ and $a \in \mathbb{F}\backslash\{0, 1\}$ such that $V = V_d(a)$. Let $\langle\, ,\, \rangle$ denote the bilinear form on the $\mathfrak{sl}_2$-module V_d from Lemma 7.8. This form meets the requirements of the present lemma since each element of $\boxtimes$ acts on $V_d(a)$ as an element of $\mathfrak{sl}_2$. This shows that the required bilinear form exists. The remaining assertions follow from Lemma 7.8 and since $EV_a : \boxtimes \to \mathfrak{sl}_2$ is surjective. $\square$

Definition 7.10 Referring to Lemma 7.9 we call $\langle\, ,\, \rangle$ a *standard bilinear form* for V.

Lemma 7.11 *Let V denote an evaluation module for $\boxtimes$. Then for $\sigma \in S_4$ the following are the same:*

(i) a standard bilinear form for V;

(ii) a standard bilinear form for V twisted via σ.

Proof: Combine Definition 6.1 and Lemma 7.9. $\square$

Lemma 7.12 *Let V denote an evaluation module for $\boxtimes$ with standard bilinear form $\langle\, ,\, \rangle$. Then for distinct $i, j \in \mathbb{I}$ the decompositions $[i, j]$ and $[j, i]$ of V are dual with respect to $\langle\, ,\, \rangle$.*

Proof: Let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ and note that $\{V_{d-n}\}_{n=0}^d$ is the decomposition $[j, i]$. For $0 \leq r, s \leq d$ with $r+s \neq d$ we show V_r, V_s are orthogonal with respect to $\langle\, ,\, \rangle$. For $u \in V_r$ and $v \in V_s$ we have $\langle x_{ij}.u, v\rangle = -\langle u, x_{ij}.v\rangle$; evaluating this using $x_{ij}.u = (2r-d)u$ and $x_{ij}.v = (2s-d)v$ we obtain $(r+s-d)\langle u, v\rangle = 0$. We assume $r+s \neq d$ so $\langle u, v\rangle = 0$ and therefore V_r, V_s are orthogonal with respect to $\langle\, ,\, \rangle$. The result follows. $\square$

Lemma 7.13 *Let $V_d(a)$ denote an evaluation module for $\boxtimes$ with standard bilinear form $\langle\, ,\, \rangle$. Then for $i \in \mathbb{I}$ and for $0 \leq n \leq d-1$ the following are orthogonal complements with respect to $\langle\, ,\, \rangle$:*

(i) component n of the flag $[i]$;

(ii) component $d-n-1$ of the flag $[i]$.

Proof: Fix $j \in \mathbb{I}$ $(j \neq i)$ and let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ of $V_d(a)$. By construction, component n (resp. component $d-n-1$) of the flag $[i]$ is $V_0 + V_1 + \cdots + V_n$ (resp. $V_0 + V_1 + \cdots + V_{d-n-1}$). These components are orthogonal by Lemma 7.12. Moreover the sum of their dimensions is $d+1$ and this equals the dimension of $V_d(a)$. Therefore these components are orthogonal complements. $\square$

Proposition 7.14 *Let V denote an evaluation module for $\boxtimes$ with standard bilinear form $\langle\, ,\, \rangle$. Pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider an $[i, j, k, \ell]$-basis for V from Definition 7.1. Denoting this basis by $\{u_n\}_{n=0}^d$ we have*

$$\langle u_r, u_s\rangle = \delta_{r+s,d}(-1)^r \binom{d}{r} \langle u_0, u_d\rangle \tag{30}$$

for $0 \leq r, s \leq d$.

Proof: If $r+s \neq d$ then $\langle u_r, u_s\rangle = 0$ by Lemma 7.12. Also by Lemma 7.9 we have

$$\langle x_{ki}.u_n, u_{d-n+1}\rangle = -\langle u_n, x_{ki}.u_{d-n+1}\rangle \tag{31}$$

for $1 \leq n \leq d$. The action of x_{ki} on $\{u_n\}_{n=0}^d$ is given in Theorem 7.7. Evaluating (31) using this data we find

$$(d-n+1)\langle u_{n-1}, u_{d-n+1}\rangle = -n\langle u_n, u_{d-n}\rangle \qquad (1 \leq n \leq d).$$

Solving this recursion we find

$$\langle u_r, u_{d-r}\rangle = (-1)^r \binom{d}{r} \langle u_0, u_d\rangle \qquad (0 \leq r \leq d).$$

The result follows. $\square$

For convenience we introduce a normalization for our 24 bases.

Notation 7.15 We fix an integer $d \geq 1$ and a scalar $a \in \mathbb{F}\backslash\{0, 1\}$. We consider the $\boxtimes$-module $V = V_d(a)$ from Definition 4.11. For $i \in \mathbb{I}$ we fix a nonzero vector η_i in component 0 of the flag $[i]$ for V. Let $\langle\ ,\ \rangle$ denote a standard bilinear form on V.

Lemma 7.16 *With reference to Notation 7.15, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists a unique basis $\{u_n\}_{n=0}^d$ of V such that:*

(i) for $0 \leq n \leq d$ the vector u_n is contained in component n of the decomposition $[k, \ell]$ for V;

(ii) $\eta_i = \sum_{n=0}^d u_n$.

We denote this basis by $[i, j, k, \ell]$.

Proof: By Proposition 7.5 there exists an $[i, j, k, \ell]$-basis for V. Denote this basis by $\{u'_n\}_{n=0}^d$. By Definition 7.1, for $0 \leq n \leq d$ the vector u'_n is in component n of the decomposition $[k, \ell]$. Also $\sum_{n=0}^d u'_n$ lies in component 0 of the flag $[i]$. Define $\eta = \sum_{n=0}^d u'_n$ and note that $\eta \neq 0$. Each of η, η_i span component 0 of the flag $[i]$ so there exists a nonzero $\beta \in \mathbb{F}$ such that $\eta = \beta\eta_i$. Define $u_n = \beta^{-1}u'_n$ for $0 \leq n \leq d$. By construction $\{u_n\}_{n=0}^d$ is a basis for V that satisfies (i), (ii) above. This shows that the required basis exists. This basis is unique by condition (ii) above and Lemma 7.2. $\square$

Lemma 7.17 *With reference to Notation 7.15 the following hold:*

(i) $\langle \eta_i, \eta_i\rangle = 0$ for $i \in \mathbb{I}$;

(ii) $\langle \eta_i, \eta_j\rangle \neq 0$ for distinct $i, j \in \mathbb{I}$.

Proof: (i) The vector η_i is in component 0 of the flag $[i]$ on V. This component is orthogonal to itself by Lemma 7.13 and since $d \geq 1$.
(ii) Let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ of V. By construction η_i spans V_0 and η_j spans V_d. By Lemma 7.13 the orthogonal complement of V_0 is $V_0 + \cdots + V_{d-1}$. Therefore V_0, V_d are not orthogonal and this implies $\langle \eta_i, \eta_j\rangle \neq 0$. $\square$

Lemma 7.18 *With reference to Notation 7.15, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the basis $[i, j, k, \ell]$ for V from Lemma 7.16. Denoting this basis by $\{u_n\}_{n=0}^d$ we have:*

(i) $u_0 = \eta_k \frac{\langle \eta_i, \eta_\ell \rangle}{\langle \eta_k, \eta_\ell \rangle}$;

(ii) $u_d = \eta_\ell \frac{\langle \eta_k, \eta_i \rangle}{\langle \eta_k, \eta_\ell \rangle}$.

Proof: (i) By construction there exists $\gamma \in \mathbb{F}$ such that $u_0 = \gamma \eta_k$. We show $\gamma = \langle \eta_i, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$. By Lemma 7.12 we have $\langle u_n, \eta_\ell \rangle = 0$ for $1 \le n \le d$. By this and Lemma 7.16(ii),

$$\langle \eta_i, \eta_\ell \rangle = \langle u_0 + \cdots + u_d, \eta_\ell \rangle = \langle u_0, \eta_\ell \rangle = \gamma \langle \eta_k, \eta_\ell \rangle.$$

Therefore $\gamma = \langle \eta_i, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$.
(ii) Similar to the proof of (i) above. $\square$

Proposition 7.19 *With reference to Notation 7.15, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the basis $[i, j, k, \ell]$ for V from Lemma 7.16. Denoting this basis by $\{u_n\}_{n=0}^d$ we have*

$$\langle u_r, u_s \rangle = \delta_{r+s,d} (-1)^r \binom{d}{r} \frac{\langle \eta_k, \eta_i \rangle \langle \eta_i, \eta_\ell \rangle}{\langle \eta_k, \eta_\ell \rangle} \tag{32}$$

for $0 \le r, s \le d$.

Proof: In line (30) evaluate $\langle u_0, u_d \rangle$ using Lemma 7.18. $\square$

In Lemma 7.16 we defined 24 bases for $V_d(a)$. We now compute the transition matrices between certain pairs of bases among these 24. First we recall a few terms.

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Suppose we are given two bases for V, written $\{u_n\}_{n=0}^d$ and $\{v_n\}_{n=0}^d$. By the *transition matrix* from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$ we mean the matrix $T \in \mathrm{Mat}_{d+1}(\mathbb{F})$ such that

$$v_j = \sum_{i=0}^{d} T_{ij} u_i \qquad (0 \le j \le d). \tag{33}$$

We recall a few properties of transition matrices. Let T denote the transition matrix from $\{u_n\}_{n=0}^d$ to $\{v_n\}_{n=0}^d$. Then T^{-1} exists, and equals the transition matrix from $\{v_n\}_{n=0}^d$ to $\{u_n\}_{n=0}^d$. Let $\{w_n\}_{n=0}^d$ denote a basis for V, and let S denote the transition matrix from $\{v_n\}_{n=0}^d$ to $\{w_n\}_{n=0}^d$. Then TS is the transition matrix from $\{u_n\}_{n=0}^d$ to $\{w_n\}_{n=0}^d$. For a linear transformation $A : V \to V$ let H (resp. K) denote the matrix representing A with respect to $\{u_n\}_{n=0}^d$ (resp. $\{v_n\}_{n=0}^d$). Then $HT = TK$.

The following matrix will play a role in our discussion. For an integer $d \ge 0$ let $Z = Z(d)$ denote the matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$ with entries

$$Z_{ij} = \begin{cases} 1, & \text{if } \ i + j = d; \\ 0, & \text{if } \ i + j \ne d \end{cases} \qquad (0 \le i, j \le d). \tag{34}$$

We observe $Z^2 = I$.

Theorem 7.20 *With reference to Notation 7.15 and Lemma 7.16, pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and consider the transition matrices from the basis $[i, j, k, \ell]$ to the bases*

$$[j, i, k, \ell], \qquad [i, k, j, \ell], \qquad [i, j, \ell, k].$$

(i) The first transition matrix is diagonal with (r, r)-entry

$$\frac{\langle \eta_j, \eta_\ell \rangle}{\langle \eta_i, \eta_\ell \rangle} \alpha^r$$

for $0 \le r \le d$, where α is the (i, j, k, ℓ)-relative of a from Definition 6.11.

(ii) The second transition matrix is lower triangular with (r, s)-entry

$$\binom{r}{s} \alpha^{r-s} (1 - \alpha)^s$$

for $0 \le s \le r \le d$, where α is the (i, j, k, ℓ)-relative of a from Definition 6.11.

(iii) The third transition matrix is the matrix Z from (34).

Proof: (i) Let T denote the transition matrix from $[i, j, k, \ell]$ to $[j, i, k, \ell]$. Then T is diagonal by Lemma 7.16(i). Let H (resp. K) denote the matrix that represents $x_{j\ell}$ with respect to $[i, j, k, \ell]$ (resp. $[j, i, k, \ell]$). Then $HT = TK$. By Theorem 7.7, H is lower bidiagonal with entries $H_{nn} = 2n - d$ for $0 \le n \le d$ and $H_{n,n-1} = -2\alpha n$ for $1 \le n \le d$. Similarly K is lower bidiagonal with entries $K_{nn} = 2n - d$ for $0 \le n \le d$ and $K_{n,n-1} = -2n$ for $1 \le n \le d$. Evaluating $HT = TK$ using these comments we find $T_{nn} = \alpha T_{n-1,n-1}$ for $1 \le n \le d$. Therefore

$$T_{rr} = T_{00} \alpha^r \qquad (0 \le r \le d). \tag{35}$$

By Lemma 7.18(i) the 0th component of the basis $[i, j, k, \ell]$ is $\eta_k \langle \eta_i, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$. Also by Lemma 7.18(i), the 0th component of the basis $[j, i, k, \ell]$ is $\eta_k \langle \eta_j, \eta_\ell \rangle \langle \eta_k, \eta_\ell \rangle^{-1}$. By these comments the 0th component of the basis $[j, i, k, \ell]$ is $\langle \eta_j, \eta_\ell \rangle \langle \eta_i, \eta_\ell \rangle^{-1}$ times the 0th component of the basis $[i, j, k, \ell]$. Therefore $T_{00} = \langle \eta_j, \eta_\ell \rangle \langle \eta_i, \eta_\ell \rangle^{-1}$. Combining this with (35) we get the result.

(ii) Let T denote the transition matrix from $[i, j, k, \ell]$ to $[i, k, j, \ell]$. For each of these two bases the sum of the vectors is η_i, so T has constant row sum 1. By construction T is lower triangular. Let H (resp. K) denote the matrix that represents $x_{\ell j}$ with respect to $[i, j, k, \ell]$ (resp. $[i, k, j, \ell]$). Then $HT = TK$. By Theorem 7.7, H is lower bidiagonal with entries $H_{nn} = d - 2n$ for $0 \le n \le d$ and $H_{n,n-1} = 2\alpha n$ for $1 \le n \le d$. Also K is diagonal with entries $K_{nn} = d - 2n$ for $0 \le n \le d$. Evaluating $HT = TK$ using these comments we find $(r - s)T_{rs} = \alpha r T_{r-1,s}$ for $0 \le s < r \le d$. By this recursion,

$$T_{rs} = \binom{r}{s} \alpha^{r-s} T_{ss} \qquad (0 \le s \le r \le d). \tag{36}$$

By (36) and since T has constant row sum 1 we routinely obtain $T_{ss} = (1 - \alpha)^s$ for $0 \le s \le d$ by induction on s. The result follows.

(iii) Immediate from Lemma 7.16. $\square$

Lemma 7.21 *With reference to Notation 7.15, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ we have*

$$\frac{\langle \eta_i, \eta_\ell \rangle}{\langle \eta_i, \eta_k \rangle} \frac{\langle \eta_j, \eta_k \rangle}{\langle \eta_j, \eta_\ell \rangle} = \alpha^d,$$

where α is the (i, j, k, ℓ)-relative of a from Definition 6.11.

Proof: Let T (resp. T') denote the transition matrix from $[i, j, k, \ell]$ to $[j, i, k, \ell]$ (resp. $[j, i, \ell, k]$ to $[i, j, \ell, k]$). By Theorem 7.20(iii), the matrix Z is the transition matrix from $[j, i, k, \ell]$ to $[j, i, \ell, k]$, and from $[i, j, \ell, k]$ to $[i, j, k, \ell]$. Therefore $TZT'Z = I$. By Theorem 7.20(i), T is diagonal with entries $T_{nn} = \langle \eta_j, \eta_\ell \rangle \langle \eta_i, \eta_\ell \rangle^{-1} \alpha^n$ for $0 \leq n \leq d$. By Lemma 6.12(i),(iii) the (j, i, ℓ, k)-relative of a coincides with the (i, j, k, ℓ)-relative of a and is therefore equal to α. By this and Theorem 7.20(i), T' is diagonal with entries $T'_{nn} = \langle \eta_i, \eta_k \rangle \langle \eta_j, \eta_k \rangle^{-1} \alpha^n$ for $0 \leq n \leq d$. Evaluating $TZT'Z = I$ using these comments we obtain the result. $\square$

Corollary 7.22 *With reference to Notation 7.15 we have*

$$\begin{aligned} \frac{\langle \eta_0, \eta_1 \rangle}{\langle \eta_0, \eta_3 \rangle} \frac{\langle \eta_2, \eta_3 \rangle}{\langle \eta_2, \eta_1 \rangle} &= a^d, \\ \frac{\langle \eta_0, \eta_2 \rangle}{\langle \eta_0, \eta_1 \rangle} \frac{\langle \eta_3, \eta_1 \rangle}{\langle \eta_3, \eta_2 \rangle} &= (1 - a^{-1})^d, \\ \frac{\langle \eta_0, \eta_3 \rangle}{\langle \eta_0, \eta_2 \rangle} \frac{\langle \eta_1, \eta_2 \rangle}{\langle \eta_1, \eta_3 \rangle} &= (1 - a)^{-d}. \end{aligned}$$

Proof: Use Lemma 7.21 and Proposition 6.14. $\square$

Note 7.23 By Corollary 7.22 and the symmetry/antisymmetry of $\langle\, , \,\rangle$ the scalars

$$\langle \eta_i, \eta_j \rangle \qquad i, j \in \mathbb{I}, \quad i \neq j$$

are determined by the sequence

$$\langle \eta_0, \eta_1 \rangle, \qquad \langle \eta_0, \eta_2 \rangle, \qquad \langle \eta_0, \eta_3 \rangle, \qquad \langle \eta_1, \eta_2 \rangle. \tag{37}$$

The scalars (37) are "free" in the following sense. Given a sequence Ψ of four nonzero scalars in $\mathbb{F}$, there exist vectors η_i ($i \in \mathbb{I}$) as in Notation 7.15 such that the sequence (37) is equal to Ψ.

8 Realizing the evaluation modules for $\boxtimes$ using polynomials in two variables

In this section we give a concrete realization of the evaluation modules for $\boxtimes$ using polynomials in two variables.

Notation 8.1 Let z_0, z_1 denote commuting indeterminates. Let $\mathbb{F}[z_0, z_1]$ denote the $\mathbb{F}$-algebra of all polynomials in z_0, z_1 that have coefficients in $\mathbb{F}$. We abbreviate $\mathcal{P} = \mathbb{F}[z_0, z_1]$ and often view this as a vector space over $\mathbb{F}$. For an integer $d \geq 0$ let $\mathcal{P}_d$ denote the subspace of $\mathcal{P}$ consisting of the homogeneous polynomials in z_0, z_1 that have total degree d. Thus $\{z_0^{d-n} z_1^n\}_{n=0}^d$ is a basis for $\mathcal{P}_d$. Note that

$$\mathcal{P} = \sum_{n=0}^{\infty} \mathcal{P}_d \qquad \text{(direct sum)} \tag{38}$$

and that $\mathcal{P}_r \mathcal{P}_s = \mathcal{P}_{r+s}$ for $r, s \geq 0$. We fix mutually distinct $\beta_i \in \mathbb{F}$ $(i \in \mathbb{I})$. Then there exist unique $z_2, z_3 \in \mathcal{P}$ such that

$$\sum_{i \in \mathbb{I}} z_i = 0, \qquad \sum_{i \in \mathbb{I}} \beta_i z_i = 0. \tag{39}$$

For the following three lemmas the proofs are routine and left to the reader.

Lemma 8.2 *With reference to Notation 8.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ we have*

$$\begin{aligned} z_k &= \frac{\beta_\ell - \beta_i}{\beta_k - \beta_\ell} z_i + \frac{\beta_\ell - \beta_j}{\beta_k - \beta_\ell} z_j, \\ z_\ell &= \frac{\beta_i - \beta_k}{\beta_k - \beta_\ell} z_i + \frac{\beta_j - \beta_k}{\beta_k - \beta_\ell} z_j. \end{aligned}$$

Lemma 8.3 *With reference to Notation 8.1, for distinct $i, j \in \mathbb{I}$ the elements*

$$z_i^r z_j^s \qquad 0 \leq r, s < \infty$$

form a basis for $\mathcal{P}$.

Lemma 8.4 *With reference to Notation 8.1, for an integer $d \geq 0$ and distinct $i, j \in \mathbb{I}$ the elements $\{z_i^{d-n} z_j^n\}_{n=0}^d$ form a basis for $\mathcal{P}_d$.*

Our next goal is to display a $\boxtimes$-module structure on $\mathcal{P}$.

Definition 8.5 [16, p. 4] With reference to Notation 8.1, by a *derivation* of $\mathcal{P}$ we mean an $\mathbb{F}$-linear map $D : \mathcal{P} \to \mathcal{P}$ such that $D(uv) = D(u)v + uD(v)$ for $u, v \in \mathcal{P}$.

Lemma 8.6 *With reference to Notation 8.1 let D denote a derivation of $\mathcal{P}$. Then $D(z^r) = rz^{r-1}D(z)$ for $z \in \mathcal{P}$ and $r \geq 1$. Moreover $D(1) = 0$.*

Proof: The first equation is a routine consequence of Definition 8.5. Setting $z = 1$ and $r = 2$ in this equation we find $D(1) = 2D(1)$ so $D(1) = 0$. $\square$

Lemma 8.7 *With reference to Notation 8.1, a derivation of $\mathcal{P}$ is 0 if and only if it vanishes on $\mathcal{P}_1$.*

Proof: Let D denote the derivation in question. We assume D vanishes on $\mathcal{P}_1$ and show $D = 0$. For distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ we have $D(z_i^r z_j^s) = 0$ by Definition 8.5 and Lemma 8.6. Now $D = 0$ in view of Lemma 8.3. $\square$

Proposition 8.8 *With reference to Notation 8.1, there exists a unique $\boxtimes$-module structure on $\mathcal{P}$ such that:*

(i) each element of $\boxtimes$ acts as a derivation on $\mathcal{P}$;

(ii) $x_{ij}.z_i = -z_i$ and $x_{ij}.z_j = z_j$ for distinct $i, j \in \mathbb{I}$.

Proof: Let $\operatorname{Der}\mathcal{P}$ denote the set of derivations for $\mathcal{P}$. Recall that $\operatorname{Der}\mathcal{P}$ is a Lie algebra over $\mathbb{F}$ with Lie bracket $[D, D'] = DD' - D'D$. For distinct $i, j \in \mathbb{I}$ define an $\mathbb{F}$-linear map $D_{ij} : \mathcal{P} \to \mathcal{P}$ such that

$$D_{ij}(z_i^r z_j^s) = (s - r) z_i^r z_j^s \qquad\qquad r, s = 0, 1, 2, \ldots$$

One checks that D_{ij} is the unique element of $\operatorname{Der}\mathcal{P}$ that sends $z_i \mapsto -z_i$ and $z_j \mapsto z_j$. Next one checks that the maps $\{D_{ij} | i, j \in \mathbb{I}, i \neq j\}$ satisfy the defining relations for $\boxtimes$ given in Definition 2.2. To do this, it suffices to verify that these relations hold on $\mathcal{P}_1$, in view of Lemma 8.7. From our comments so far, there exists a Lie algebra homomorphism from $\boxtimes$ to $\operatorname{Der}\mathcal{P}$ that sends $x_{ij} \mapsto D_{ij}$ for all distinct $i, j \in \mathbb{I}$. This shows that the required $\boxtimes$-module structure exists. This $\boxtimes$-module structure is unique in view of Lemma 8.7. $\square$

We emphasize the following.

Lemma 8.9 *With reference to Notation 8.1 and Proposition 8.8, for distinct $i, j \in \mathbb{I}$ and integers $r, s \geq 0$ the element $z_i^r z_j^s$ is an eigenvector for x_{ij} with eigenvalue $s - r$.*

Lemma 8.10 *With reference to Notation 8.1,*

(i) for $d \geq 0$ the subspace $\mathcal{P}_d$ is an irreducible $\boxtimes$-submodule of $\mathcal{P}$;

(ii) the $\boxtimes$-module $\mathcal{P}_0$ is trivial;

(iii) for $d \geq 1$ the $\boxtimes$-module $\mathcal{P}_d$ is isomorphic to an evaluation module.

Proof: Pick distinct $i, j \in \mathbb{I}$ and consider the basis $\{z_i^{d-n} z_j^n\}_{n=0}^d$ for $\mathcal{P}_d$ from Lemma 8.4. By Lemma 8.9, for $0 \leq n \leq d$ the vector $z_i^{d-n} z_j^n$ is an eigenvector for x_{ij} with eigenvalue $2n - d$. Therefore $\mathcal{P}_d$ is invariant under x_{ij}. We have now shown that $\mathcal{P}_d$ is a $\boxtimes$-submodule. This module is irreducible since the eigenvalues of x_{ij} on $\mathcal{P}_d$ are $2n - d$ $(0 \leq n \leq d)$ and the corresponding eigenspaces have dimension 1. This proves (i). To get (ii) note that $\mathcal{P}_0$ has dimension 1. To get (iii) apply Proposition 5.4. $\square$

We now find the evaluation parameter for the $\boxtimes$-modules given in Lemma 8.10(iii).

Lemma 8.11 *With reference to Notation 8.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the following vanishes on the $\boxtimes$-module $\mathcal{P}$:*

$$(\beta_i - \beta_j)(\beta_k - \beta_\ell)x_{ij} + (\beta_i - \beta_k)(\beta_\ell - \beta_j)x_{ik} + (\beta_i - \beta_\ell)(\beta_j - \beta_k)x_{i\ell}. \qquad (40)$$

Proof: Using Lemma 8.2 and Proposition 8.8(ii) we routinely find that the derivation (40) vanishes on each of z_i, z_j and hence on $\mathcal{P}_1$. This derivation is 0 in view of Lemma 8.7. $\square$

Theorem 8.12 *With reference to Notation 8.1, for an integer $d \geq 1$ the $\boxtimes$-module $\mathcal{P}_d$ is isomorphic to $V_d(a)$ where*

$$a = \frac{\beta_0 - \beta_1}{\beta_0 - \beta_3} \frac{\beta_2 - \beta_3}{\beta_2 - \beta_1}.$$

Proof: We invoke Proposition 5.1 with $V = \mathcal{P}_d$. Note that $a \notin \{0, 1\}$ since the β_i $(i \in \mathbb{I})$ are mutually distinct. By Lemma 8.11, the four expressions in (17), (18) vanish on $\mathcal{P}$ and in particular on $\mathcal{P}_d$. By this and Proposition 5.1 the $\boxtimes$-module $\mathcal{P}_d$ has evaluation parameter a. Now the $\boxtimes$-module $\mathcal{P}_d$ is isomorphic to $V_d(a)$ since $\mathcal{P}_d$ has dimension $d + 1$. $\square$

Earlier in the paper we described the $\boxtimes$-module $V_d(a)$. We now consider how things look from the point of view of $\mathcal{P}_d$.

Proposition 8.13 *With reference to Notation 8.1, for an integer $d \geq 0$ and distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ on $\mathcal{P}_d$ is described as follows: for $0 \leq n \leq d$ the nth component is spanned by $z_i^{d-n} z_j^n$.*

Proof: By Lemma 8.4 the vectors $\{z_i^{d-n} z_j^n\}_{n=0}^d$ form a basis for $\mathcal{P}_d$. By Lemma 8.9, for $0 \leq n \leq d$ the vector $z_i^{d-n} z_j^n$ is an eigenvector for x_{ij} with eigenvalue $2n - d$. The result follows. $\square$

Proposition 8.14 *With reference to Notation 8.1, for an integer $d \geq 0$ and $i \in \mathbb{I}$ the flag $[i]$ on $\mathcal{P}_d$ is described as follows: for $0 \leq n \leq d$ the nth component is $z_i^{d-n} \mathcal{P}_n$.*

Proof: Let U_n denote the component in question and pick $j \in \mathbb{I}, j \neq i$. By construction U_n is the sum of components $0, 1, \ldots, n$ for the decomposition $[i, j]$ of $\mathcal{P}_d$. By this and Proposition 8.13, U_n has a basis $\{z_i^{d-r} z_j^r\}_{r=0}^n$. Note that $\{z_i^{n-r} z_j^r\}_{r=0}^n$ is a basis for $\mathcal{P}_n$ so $\{z_i^{d-r} z_j^r\}_{r=0}^n$ is a basis for $z_i^{d-n} \mathcal{P}_n$. Therefore $U_n = z_i^{d-n} \mathcal{P}_n$. $\square$

Definition 8.15 *With reference to Notation 8.1, for an integer $d \geq 1$ and $i \in \mathbb{I}$ the element z_i^d is in component 0 of the flag $[i]$ on $\mathcal{P}_d$. In Notation 7.15 we let η_i denote any nonzero element in this component. For the rest of this section we choose $\eta_i = z_i^d$.*

Proposition 8.16 *With reference to Notation 8.1, for an integer $d \geq 1$ and for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the basis $[i, j, k, \ell]$ of $\mathcal{P}_d$ is described as follows. For $0 \leq n \leq d$ the nth component is*

$$z_k^{d-n} z_\ell^n \binom{d}{n} \frac{(\beta_j - \beta_k)^{d-n}(\beta_j - \beta_\ell)^n}{(\beta_i - \beta_j)^d}. \tag{41}$$

Proof: Abbreviate u_n for the polynomial (41) and note that $\{u_n\}_{n=0}^d$ is a basis for $\mathcal{P}_d$ by Lemma 8.4. We show that this basis satisfies Lemma 7.16(i),(ii). By Proposition 8.13, for $0 \leq n \leq d$ the vector u_n is in component n of the decomposition $[k, \ell]$ for $\mathcal{P}_d$. Also $\eta_i = z_i^d$ by Definition 8.15, and

$$z_i^d \;=\; \Big(\frac{\beta_j - \beta_k}{\beta_i - \beta_j} z_k + \frac{\beta_j - \beta_\ell}{\beta_i - \beta_j} z_\ell\Big)^d \;=\; \sum_{n=0}^{d} u_n$$

by Lemma 8.2 and the binomial theorem. Therefore the basis $\{u_n\}_{n=0}^d$ satisfies Lemma 7.16(i),(ii) and the result follows. $\square$

Lemma 8.17 *With reference to Notation 8.1 and Theorem 8.12, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ the (i, j, k, ℓ)-relative of a is*

$$\frac{\beta_i - \beta_\ell}{\beta_i - \beta_k} \frac{\beta_j - \beta_k}{\beta_j - \beta_\ell}.$$

Proof: Very similar to the proof of Lemma 6.13. $\square$

We now consider the standard bilinear form.

Lemma 8.18 *With reference to Notation 8.1, for an integer $d \geq 1$ there exists a standard bilinear form $\langle\, , \rangle$ on the $\boxtimes$-module $\mathcal{P}_d$ that satisfies the following. For distinct $i, j \in \mathbb{I}$,*

$$\langle z_i^d, z_j^d \rangle = (\beta_k - \beta_\ell)^d$$

where the set $\{k, \ell\}$ is the complement of $\{i, j\}$ in $\mathbb{I}$, and the pair k, ℓ is ordered so that the sequence (i, j, k, ℓ) is sent to $(2, 0, 1, 3)$ by an even permutation in S_4.

Proof: For the time being let $\langle\, , \rangle$ denote any standard bilinear form on $\mathcal{P}_d$. For mutually distinct $i, j, k, \ell \in \mathbb{I}$ define

$$\tau(i, j, k, \ell) = \frac{\langle z_i^d, z_j^d \rangle}{(\beta_k - \beta_\ell)^d},$$

and note that $\tau(i, j, k, \ell) \neq 0$ by Lemma 7.17(ii). We claim

- $\tau(j, i, k, \ell) = (-1)^d \tau(i, j, k, \ell)$,
- $\tau(i, k, j, \ell) = (-1)^d \tau(i, j, k, \ell)$,

- $\tau(i,j,\ell,k) = (-1)^d \tau(i,j,k,\ell)$.

The first bullet follows from the symmetry/antisymmetry of $\langle\,,\,\rangle$ in Lemma 7.9. The third bullet follows from the construction. To verify the second bullet we show

$$\frac{\langle z_i^d, z_k^d\rangle}{(\beta_\ell - \beta_j)^d} = \frac{\langle z_i^d, z_j^d\rangle}{(\beta_k - \beta_\ell)^d}. \qquad (42)$$

To verify (42), in the left-hand side eliminate z_k using the first equation of Lemma 8.2, and note that $\langle z_i^d, z_i^{d-n} z_j^n\rangle = 0$ for $0 \leq n \leq d-1$ in view of Lemma 7.12 and Proposition 8.13. This proves (42) and we have now verified the three bullets. Multiplying $\langle\,,\,\rangle$ by a nonzero scalar in $\mathbb{F}$ if necessary, we may assume $\tau(2,0,1,3) = 1$. For $\sigma \in S_4$ define $\mathrm{sgn}(\sigma)$ to be 1 if σ is even and -1 if σ is odd. Using the three bullets and $\tau(2,0,1,3) = 1$ we obtain

$$\tau(i,j,k,\ell) = \mathrm{sgn}(\sigma)^d$$

where $\sigma \in S_4$ sends the sequence (i,j,k,ℓ) to $(2,0,1,3)$. The result follows. $\square$

Corollary 8.19 *Referring to the bilinear form $\langle\,,\,\rangle$ in Lemma 8.18, for $i,j \in \mathbb{I}$ the scalar $\langle z_i^d, z_j^d\rangle$ is displayed in row i, column j of the table below.*

	0	1	2	3
0	0	$(\beta_2-\beta_3)^d$	$(\beta_3-\beta_1)^d$	$(\beta_1-\beta_2)^d$
1	$(\beta_3-\beta_2)^d$	0	$(\beta_0-\beta_3)^d$	$(\beta_2-\beta_0)^d$
2	$(\beta_1-\beta_3)^d$	$(\beta_3-\beta_0)^d$	0	$(\beta_0-\beta_1)^d$
3	$(\beta_2-\beta_1)^d$	$(\beta_0-\beta_2)^d$	$(\beta_1-\beta_0)^d$	0

Proof: Routine using Lemma 7.17(i) and Lemma 8.18. $\square$

Theorem 8.20 *With reference to Notation 8.1, for an integer $d \geq 1$ consider the standard bilinear form $\langle\,,\,\rangle$ on $\mathcal{P}_d$ from Lemma 8.18. For distinct $i,j \in \mathbb{I}$ and $0 \leq r,s \leq d$ we have*

$$\langle z_i^{d-r} z_j^r, z_i^{d-s} z_j^s\rangle = \delta_{r+s,d}(-1)^r \binom{d}{r}^{-1} (\beta_k - \beta_\ell)^d \qquad (43)$$

where the set $\{k,\ell\}$ is the complement of $\{i,j\}$ in $\mathbb{I}$, and the pair k,ℓ is ordered so that the sequence (i,j,k,ℓ) is sent to $(2,0,1,3)$ by an even permutation in S_4.

Proof: Assume $r+s=d$; otherwise (43) holds by Lemma 7.12 and Proposition 8.13. Let $\{u_n\}_{n=0}^d$ denote the basis $[k,\ell,i,j]$ for $\mathcal{P}_d$. Using Proposition 7.14 and Proposition 8.16 we find

$$\begin{aligned}(-1)^r\binom{d}{r} &= \frac{\langle u_r, u_{d-r}\rangle}{\langle u_0, u_d\rangle} \\ &= \frac{\langle z_i^{d-r} z_j^r, z_i^r z_j^{d-r}\rangle}{\langle z_i^d, z_j^d\rangle}\binom{d}{r}^2.\end{aligned}$$

Line (43) follows in view of Lemma 8.18. $\square$

Recall the subgroup G of S_4 from Definition 6.8. In Corollary 6.10 we showed that for an evaluation module of $\boxtimes$ and for $\sigma \in G$, twisting the module via σ does not change the isomorphism class of the module. We now interpret this fact using the $\boxtimes$-module $\mathcal{P}$.

With reference to Notation 8.1, recall an *automorphism* of $\mathcal{P}$ is an $\mathbb{F}$-linear bijection $\phi : \mathcal{P} \to \mathcal{P}$ such that $\phi(uv) = \phi(u)\phi(v)$ for $u, v \in \mathcal{P}$.

Lemma 8.21 *With reference to Notation 8.1, for an automorphism ϕ of $\mathcal{P}$ and a derivation D of $\mathcal{P}$ the composition $\phi^{-1}D\phi$ is a derivation of $\mathcal{P}$.*

Proof: Routine using Definition 8.5 and the definition of automorphism. $\square$

Lemma 8.22 *With reference to Notation 8.1, for mutually distinct $i, j, k, \ell \in \mathbb{I}$ there exists a unique automorphism of $\mathcal{P}$ that sends*

$$z_i \mapsto \frac{\beta_j - \beta_k}{\beta_i - \beta_k} z_j, \qquad\qquad z_j \mapsto \frac{\beta_i - \beta_\ell}{\beta_j - \beta_\ell} z_i, \tag{44}$$

$$z_k \mapsto \frac{\beta_\ell - \beta_i}{\beta_i - \beta_k} z_\ell, \qquad\qquad z_\ell \mapsto \frac{\beta_k - \beta_j}{\beta_j - \beta_\ell} z_k. \tag{45}$$

Proof: Since z_i, z_j form a basis for $\mathcal{P}_1$ there exists an $\mathbb{F}$-linear transformation $\phi : \mathcal{P}_1 \to \mathcal{P}_1$ that satisfies (44). Observe that ϕ^{-1} exists, and that ϕ extends uniquely to an automorphism of $\mathcal{P}$. To get (45) combine Lemma 8.2 and (44). $\square$

Proposition 8.23 *With reference to Notation 8.1, for $\sigma \in G$ there exists an automorphism ϕ_σ of $\mathcal{P}$ that sends z_r into $\mathbb{F}z_{\sigma(r)}$ for all $r \in \mathbb{I}$.*

Proof: Assume σ is not the identity; otherwise the result is clear. By Definition 6.8 there exists mutually distinct $i, j, k, \ell \in \mathbb{I}$ such that $\sigma = (i, j)(k, \ell)$; let ϕ_σ denote the corresponding automorphism of $\mathcal{P}$ from Lemma 8.22. By (44), (45) the automorphism ϕ_σ sends z_r into $\mathbb{F}z_{\sigma(r)}$ for all $r \in \mathbb{I}$. $\square$

Proposition 8.24 *With reference to Notation 8.1, for $\sigma \in G$ and $\xi \in \boxtimes$ the equation*

$$\sigma(\xi) = \phi_\sigma \xi \phi_\sigma^{-1} \tag{46}$$

holds on $\mathcal{P}$.

Proof: Without loss we may assume that ξ is a generator x_{rs} of $\boxtimes$. Each side of (46) acts on $\mathcal{P}$ as a derivation, by Proposition 8.8 and Lemma 8.21. Now by Lemma 8.7, it suffices to show that (46) holds on $\mathcal{P}_1$. The elements $z_{\sigma(r)}$ and $z_{\sigma(s)}$ form a basis for $\mathcal{P}_1$. We now apply each side of (46) to $z_{\sigma(r)}$. Concerning the left-hand side,

$$\sigma(x_{rs}).z_{\sigma(r)} \;=\; x_{\sigma(r),\sigma(s)}.z_{\sigma(r)} \;=\; -z_{\sigma(r)}.$$

Concerning the right-hand side, first note by Proposition 8.23 that there exists $\gamma \in \mathbb{F}$ such that $\phi_\sigma(z_r) = \gamma z_{\sigma(r)}$. Observe $\gamma \neq 0$ since ϕ_σ is a bijection. We have

$$\phi_\sigma x_{rs} \phi_\sigma^{-1}.z_{\sigma(r)} \; = \; \gamma^{-1}\phi_\sigma x_{rs}.z_r \; = \; -\gamma^{-1}\phi_\sigma.z_r \; = \; -z_{\sigma(r)}.$$

Thus the two sides of (46) agree at $z_{\sigma(r)}$. A similar argument shows that the two sides of (46) agree at $z_{\sigma(s)}$. Now (46) holds on $\mathcal{P}_1$ and the result follows. □

Theorem 8.25 *With reference to Notation 8.1 and Proposition 8.23, for $\sigma \in G$ the map ϕ_σ is an isomorphism of ⊠-modules from $\mathcal{P}$ to $\mathcal{P}$ twisted via σ.*

Proof: This is a reformulation of Proposition 8.24. □

9 General finite-dimensional irreducible ⊠-modules

We have now completed our description of the evaluation modules for ⊠. For the rest of this paper we consider a general finite-dimensional irreducible ⊠-module. For this module we give a tensor product decomposition into evaluation modules, we give an explicit formula for the shape, and obtain a Drinfel'd polynomial. We also show that this module is isomorphic to its dual.

Let U and V denote ⊠-modules. By [16, p. 26] the vector space $U \otimes V$ has a ⊠-module structure given by

$$\xi.(u \otimes v) = (\xi.u) \otimes v + u \otimes (\xi.v) \qquad\qquad u \in U, \quad v \in V, \quad \xi \in \boxtimes. \tag{47}$$

Lemma 9.1 *The following hold for ⊠-modules U and V:*

(i) There exists an isomorphism of ⊠-modules from $U \otimes V$ to $V \otimes U$ that sends $u \otimes v \mapsto v \otimes u$ for all $u \in U$ and $v \in V$.

(ii) Assume the ⊠-module $U \otimes V$ is irreducible. Then U and V are irreducible.

Proof: (i) Routine.
(ii) The ⊠-module U is irreducible since if U' is a nonzero proper submodule of U then $U' \otimes V$ is a nonzero proper submodule of $U \otimes V$. The proof for V is similar. □

Theorem 9.2 [14, Section 1] *Every nontrivial finite-dimensional irreducible ⊠-module is isomorphic to a tensor product of evaluation modules. Two such tensor products are isomorphic if and only if one can be obtained from the other by permuting the factors in the tensor product. A tensor product of evaluation modules*

$$V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N)$$

is irreducible if and only if $a_1, a_2, \ldots, a_N$ are mutually distinct.

Proof: In [14, Theorems 1.7, 1.8] Hartwig gives an explicit bijection between (i) the set of isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules; (ii) the set of isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules that have type $(0,0)$. In [14, Theorems 1.3, 1.4, 1.6] Hartwig summarizes the classification of finite-dimensional irreducible $\mathcal{O}$-modules that have type $(0,0)$. This classification is due to Davies [9], [10]; see also Date and Roan [8]. If we interpret this classification as a classification of the finite-dimensional irreducible $\boxtimes$-modules via the above bijection, we get the present theorem. To aid in this interpretation we note that our definition of the evaluation parameter does not match Hartwig's definition of the evaluation parameter. Denoting our evaluation parameter by a and Hartwig's evaluation parameter by b we have $a = 4b(b+1)^{-2}$. $\Box$

Lemma 9.3 *Let U and V denote $\boxtimes$-modules. Then for $\sigma \in S_4$ the following coincide:*

(i) the $\boxtimes$-module ${}^{\sigma}(U \otimes V)$;

(ii) the $\boxtimes$-module ${}^{\sigma}U \otimes {}^{\sigma}V$.

Proof: Routine using Definition 6.1 and (47). $\Box$

Theorem 9.4 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module, and write V as a tensor product of evaluation modules:*

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

Then for $\sigma \in S_4$ the $\boxtimes$-module ${}^{\sigma}V$ is isomorphic to

$$V_{d_1}(\sigma(a_1)) \otimes V_{d_2}(\sigma(a_2)) \otimes \cdots \otimes V_{d_N}(\sigma(a_N)).$$

Proof: Combine Theorem 6.7 and Lemma 9.3. $\Box$

Corollary 9.5 *Let V denote a finite-dimensional irreducible $\boxtimes$-module and let the group G be as in Definition 6.8. Then for $\sigma \in G$ the following are isomorphic:*

(i) the $\boxtimes$-module V twisted via σ;

(ii) the $\boxtimes$-module V.

Proof: Combine Lemma 6.9 and Theorem 9.4. $\Box$

We now obtain an explicit formula for the shape of a finite-dimensional irreducible $\boxtimes$-module. We use the following notation. For an indeterminate λ let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra consisting of all polynomials in λ that have coefficients in $\mathbb{F}$.

Definition 9.6 Let V denote a finite-dimensional irreducible $\boxtimes$-module. We define a polynomial $S_V \in \mathbb{F}[\lambda]$ by

$$S_V = \sum_{n=0}^{d} \rho_n \lambda^n,$$

where $\{\rho_n\}_{n=0}^{d}$ is the shape of V. We call S_V the *shape polynomial* for V.

Example 9.7 *Let V denote an evaluation module for $\boxtimes$. Then*

$$S_V = 1 + \lambda + \lambda^2 + \cdots + \lambda^d$$

where d is the diameter of V.

Proof: Combine Proposition 5.4 and Definition 9.6. $\square$

Let U, V denote finite-dimensional irreducible $\boxtimes$-modules such that the $\boxtimes$-module $U \otimes V$ is irreducible. We are going to show that $S_{U\otimes V} = S_U S_V$.

Lemma 9.8 *Let U, V denote finite-dimensional irreducible $\boxtimes$-modules such that the $\boxtimes$-module $U \otimes V$ is irreducible. Let d (resp. δ) denote the diameter of U (resp. V).*

(i) The diameter of $U \otimes V$ is $d + \delta$.

(ii) For distinct $i, j \in \mathbb{I}$ the decomposition $[i, j]$ of $U \otimes V$ is described as follows. For $0 \leq n \leq d + \delta$ the nth component is

$$\sum_{r,s} U_r \otimes V_s, \tag{48}$$

where $\{U_r\}_{r=0}^{d}$ (resp. $\{V_s\}_{s=0}^{\delta}$) denotes the decomposition $[i, j]$ of U (resp. V), and where the sum is over all ordered pairs r, s such that $0 \leq r \leq d$, $0 \leq s \leq \delta$, $r + s = n$.

Proof: For $0 \leq n \leq d + \delta$ let $(U \otimes V)_n$ denote the sum (48). By (47), each element of $(U \otimes V)_n$ is an eigenvector for x_{ij} with eigenvalue $2n - d - \delta$. By construction the sequence $\{(U \otimes V)_n\}_{n=0}^{d+\delta}$ is a decomposition of $U \otimes V$. The results follow. $\square$

Corollary 9.9 *Let U, V denote finite-dimensional irreducible $\boxtimes$-modules such that the $\boxtimes$-module $U \otimes V$ is irreducible. Then with reference to Definition 9.6,*

$$S_{U\otimes V} = S_U S_V.$$

Proof: Adopt the notation of Lemma 9.8. For $0 \leq n \leq d + \delta$ the sum (48) is direct so

$$\rho_n(U \otimes V) = \sum_{r,s} \rho_r(U)\rho_s(V), \tag{49}$$

where the sum is over all ordered pairs r, s such that $0 \le r \le d$, $0 \le s \le \delta$, $r + s = n$. By Definition 9.6,

$$S_{U\otimes V} = \sum_{n=0}^{d+\delta} \rho_n(U \otimes V)\lambda^n.$$

Evaluating this using (49) we routinely find $S_{U\otimes V} = S_U S_V$. □

Theorem 9.10 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module, and write V as a tensor product of evaluation modules:*

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

Then the shape $\{\rho_n\}_{n=0}^d$ of V satisfies

$$\sum_{n=0}^{d} \rho_n \lambda^n = \prod_{j=1}^{N} (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

In particular $\rho_0 = 1$.

Proof: Combine Example 9.7 and Corollary 9.9. □

Our next goal is to obtain a Drinfel'd polynomial for each finite-dimensional irreducible $\boxtimes$-module.

Lemma 9.11 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Pick mutually distinct $i, j, k \in \mathbb{I}$ and consider the action of $x_{ij} + x_{jk}$ on the decomposition $[i, j]$ of V. Denoting this decomposition by $\{V_n\}_{n=0}^d$ we have $(x_{ij} + x_{jk})V_n \subseteq V_{n+1}$ for $0 \le n \le d$.*

Proof: By the definition of $[i, j]$,

$$(x_{ij} - (2n - d)I)V_n = 0. \tag{50}$$

By the table in Section 3,

$$(x_{jk} - (d - 2n)I)V_n \subseteq V_{n+1}. \tag{51}$$

Adding (50), (51) we get the result. □

Definition 9.12 Let V denote a finite-dimensional irreducible $\boxtimes$-module and let $\{V_n\}_{n=0}^d$ denote the decompostion $[1, 3]$ of V. Note that $\dim(V_0) = 1$ by Theorem 9.10. Abbreviate

$$e^+ := \frac{x_{13} + x_{30}}{2}, \qquad e^- := \frac{x_{31} + x_{12}}{2}. \tag{52}$$

By Lemma 9.11 and since the decomposition $[3, 1]$ is the inversion of $[1, 3]$,

$$e^+V_n \subseteq V_{n+1}, \qquad e^-V_n \subseteq V_{n-1} \qquad (0 \le n \le d). \tag{53}$$

For an integer $i \ge 0$ the space V_0 is invariant under $(e^-)^i(e^+)^i$; let $\vartheta_i = \vartheta_i(V)$ denote the corresponding eigenvalue. Note that $\vartheta_i = 0$ for $i > d$.

Definition 9.13 Let V denote a finite-dimensional irreducible $\boxtimes$-module. We define a polynomial $P_V \in \mathbb{F}[\lambda]$ by

$$P_V = \sum_{i=0}^{\infty} \frac{(-1)^i \vartheta_i \lambda^i}{(i!)^2}, \tag{54}$$

where the scalars ϑ_i are from Definition 9.12. Observe that P_V has degree at most the diameter of V. Moreover P_V has constant coefficient $\vartheta_0 = 1$. Following [5, Section 3.4], [20, Definition 4.2], [23, Section 4] we call P_V the *Drinfel'd polynomial* of V.

Lemma 9.14 *Let $V = V_d(a)$ denote an evaluation module for $\boxtimes$. Then $P_V = (1 - a\lambda)^d$.*

Proof: Let $\{V_n\}_{n=0}^d$ denote the decomposition $[1,3]$ of V. Let $\{v_n\}_{n=0}^d$ denote the basis for the $\mathfrak{sl}_2$-module V_d from Lemma 4.9. Note that v_n is a basis of V_n for $0 \le n \le d$. Using (1), (52) and Lemma 4.7 we find $EV_a(e^+) = af$ and $EV_a(e^-) = e$. By this and Lemma 4.9 we find $e^+.v_n = a(n+1)v_{n+1}$ for $0 \le n \le d-1$, $e^+.v_d = 0$, $e^-.v_n = (d-n+1)v_{n-1}$ for $1 \le n \le d$, $e^-.v_0 = 0$. Using this data we find $\vartheta_i = a^i (i!)^2 \binom{d}{i}$ for $0 \le i \le d$. Now (54) becomes

$$P_V \;=\; \sum_{i=0}^{d} \binom{d}{i} (-1)^i a^i \lambda^i \;=\; (1 - a\lambda)^d$$

by the binomial theorem. $\square$

Proposition 9.15 *Let U, V denote finite-dimensional irreducible $\boxtimes$-modules such that the $\boxtimes$-module $U \otimes V$ is irreducible. Then $P_{U \otimes V} = P_U P_V$.*

Proof: We claim that for an integer $i \ge 0$,

$$\vartheta_i(U \otimes V) = \sum_{n=0}^{i} \binom{i}{n}^2 \vartheta_{i-n}(U) \vartheta_n(V). \tag{55}$$

To prove the claim, let U_0 (resp. V_0) denote the 0th component for the decomposition $[1,3]$ of U (resp. V). By Theorem 9.10, each of U_0, V_0 has dimension 1. By Lemma 9.8(ii) the space $U_0 \otimes V_0$ is the 0th component of the decomposition $[1,3]$ of $U \otimes V$. Therefore by Definition 9.12 the scalar $\vartheta_i(U \otimes V)$ is the eigenvalue of $(e^-)^i (e^+)^i$ associated with $U_0 \otimes V_0$. Pick $0 \ne u \in U_0$ and $0 \ne v \in V_0$. Using (47) we obtain

$$\begin{aligned}
(e^-)^i (e^+)^i .(u \otimes v) &= (e^-)^i \sum_{n=0}^{i} \binom{i}{n} ((e^+)^{i-n}.u) \otimes ((e^+)^n.v) \\
&= \sum_{m=0}^{i} \sum_{n=0}^{i} \binom{i}{m} \binom{i}{n} ((e^-)^{i-m} (e^+)^{i-n}.u) \otimes ((e^-)^m (e^+)^n.v).
\end{aligned} \tag{56}$$

We examine the terms in (56). By (53) and the line below it, for $0 \le m, n \le i$ the vector $(e^-)^m (e^+)^n.v$ is equal to 0 if $m > n$ and $\vartheta_n(V)v$ if $m = n$. Similarly $(e^-)^{i-m}(e^+)^{i-n}.u$ is equal to 0 if $m < n$ and $\vartheta_{i-n}(U)u$ if $m = n$. By these comments the double sum in (56) is equal to $u \otimes v$ times the sum on the right in (55). We conclude that (55) is valid and the claim is proved. By Definition 9.13,

$$P_{U\otimes V} = \sum_{i=0}^{\infty} \frac{(-1)^i \vartheta_i(U \otimes V)\lambda^i}{(i!)^2}.$$

Evaluating this using (55) we get $P_{U\otimes V} = P_U P_V$ after a brief calculation. □

Theorem 9.16 *Let V denote a nontrivial finite-dimensional irreducible $\boxtimes$-module, and write V as a tensor product of evaluation modules:*

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

Then the Drinfel'd polynomial P_V is given by

$$P_V = \prod_{j=1}^{N} (1 - a_j \lambda)^{d_j}.$$

Proof: Combine Example 9.14 and Proposition 9.15. □

Corollary 9.17 *The map $V \mapsto P_V$ induces a bijection between the following two sets:*

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules;

(ii) the polynomials in $\mathbb{F}[\lambda]$ that have constant coefficient 1 and are nonzero at $\lambda = 1$.

Proof: Combine Theorem 9.2 and Theorem 9.16. □

Our next goal is to show that a finite-dimensional irreducible $\boxtimes$-module is isomorphic to its dual.

Theorem 9.18 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then there exists a nonzero bilinear form $\langle\, ,\, \rangle$ on V such that*

$$\langle \xi.u, v \rangle = -\langle u, \xi.v \rangle \qquad \xi \in \boxtimes, \qquad u, v \in V. \tag{57}$$

This form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. The form is nondegenerate. The form is symmetric (resp. antisymmetric) when the diameter is even (resp. odd).

Proof: Assume V is nontrivial; otherwise the proof is routine. Concerning the existence of $\langle\, ,\, \rangle$ write V as a tensor product of evaluation modules:

$$V = V_{d_1}(a_1) \otimes V_{d_2}(a_2) \otimes \cdots \otimes V_{d_N}(a_N).$$

For $1 \leq i \leq N$ let $\langle\, , \,\rangle_i$ denote a standard bilinear form on $V_{d_i}(a_i)$. Define a bilinear form $\langle\, , \,\rangle$ on V such that

$$\langle \otimes_{i=1}^{N} u_i, \otimes_{i=1}^{N} v_i \rangle = \prod_{i=1}^{N} \langle u_i, v_i \rangle_i \tag{58}$$

for all $u_i, v_i \in V_{d_i}(a_i)$ $(1 \leq i \leq N)$. By construction $\langle\, , \,\rangle$ is nonzero. Using (47) and (58) one checks that $\langle\, , \,\rangle$ satisfies (57). To show that $\langle\, , \,\rangle$ is nondegenerate, note that the subspace $\{u \in V | \langle u, v \rangle = 0 \;\forall v \in V\}$ is a proper $\boxtimes$-submodule of V and therefore zero by the irreducibility of V. Concerning the uniqueness of $\langle\, , \,\rangle$, let $\langle\, , \,\rangle'$ denote any bilinear form on V that satisfies (57). We show that $\langle\, , \,\rangle'$ is a scalar multiple of $\langle\, , \,\rangle$. Pick a basis for V, and for $\xi \in \boxtimes$ let ξ_b denote the matrix that represents ξ with respect to this basis. Let M (resp. N) denote the matrix that represents $\langle\, , \,\rangle$ (resp. $\langle\, , \,\rangle'$) with respect to the basis. Note that M is invertible since $\langle\, , \,\rangle$ is nondegenerate. By (57) we have $\xi_b^t M = -M\xi_b$ and $\xi_b^t N = -N\xi_b$ for $\xi \in \boxtimes$. Combining these equations we find that $M^{-1}N$ commutes with ξ_b for all $\xi \in \boxtimes$. Now $M^{-1}N$ is a scalar multiple of the identity by Schur's lemma [7, Lemma 27.3] and since the $\boxtimes$-module V is irreducible. By these comments N is a scalar multiple of M so $\langle\, , \,\rangle'$ is a scalar multiple of $\langle\, , \,\rangle$. We have now shown that $\langle\, , \,\rangle$ is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. To verify our assertions concerning symmetry/asymmetry, let d denote the diameter of V and note that $d = \sum_{i=1}^{N} d_i$ by Theorem 9.10. Referring to (58),

$$\begin{aligned}
\langle \otimes_{i=1}^{N} u_i, \otimes_{i=1}^{N} v_i \rangle &= \prod_{i=1}^{N} \langle u_i, v_i \rangle_i \\
&= \prod_{i=1}^{N} (-1)^{d_i} \langle v_i, u_i \rangle_i \qquad\qquad \text{(by Lemma 7.9)} \\
&= (-1)^{d} \prod_{i=1}^{N} \langle v_i, u_i \rangle_i \\
&= (-1)^{d} \langle \otimes_{i=1}^{N} v_i, \otimes_{i=1}^{N} u_i \rangle.
\end{aligned}$$

It follows that $\langle\, , \,\rangle$ is symmetric (resp. antisymmetric) when d is even (resp. d is odd). $\square$

Let V denote a finite-dimensional vector space over $\mathbb{F}$. By definition the dual space V^* is the vector space over $\mathbb{F}$ consisting of the linear transformations from V to $\mathbb{F}$. The dimensions of V and V^* coincide. Now assume that V supports a $\boxtimes$-module structure. Then V^* carries a $\boxtimes$-module structure such that for $\xi \in \boxtimes$ and $f \in V^*$,

$$(\xi.f)(v) = -f(\xi.v) \qquad\qquad v \in V. \tag{59}$$

Theorem 9.19 *Let V denote a finite-dimensional irreducible $\boxtimes$-module and let $\langle\, , \,\rangle$ denote a bilinear form on V from Theorem 9.18. Then there exists an isomorphism of $\boxtimes$-modules $\varphi : V \to V^*$ such that*

$$\varphi(u)(v) = \langle u, v \rangle \qquad\qquad u, v \in V. \tag{60}$$

Proof: By elementary linear algebra there exists a unique linear transformation $\varphi : V \to V^*$ that satisfies (60). The kernel of φ is $\{u \in V | \langle u, v \rangle = 0 \ \forall v \in V\}$. This space is zero since $\langle \, , \, \rangle$ is nondegenerate, so φ is injective. Now since V, V^* have the same dimension, the map φ is a bijection and hence an isomorphism of vector spaces. Using (57) and (59) one checks that φ is an isomorphism of $\boxtimes$-modules. $\square$

Let V denote a finite-dimensional irreducible $\boxtimes$-module. In Theorem 9.18 we displayed a bilinear form $\langle \, , \, \rangle$ on V. We now mention some related bilinear forms that are of interest.

Theorem 9.20 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For a nonidentity $\sigma \in G$ there exists a nonzero bilinear form $\langle \, , \, \rangle_\sigma$ on V such that*

$$\langle \xi.u, v \rangle_\sigma = -\langle u, \sigma(\xi).v \rangle_\sigma \qquad \xi \in \boxtimes, \qquad u, v \in V. \tag{61}$$

This form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. The form is nondegenerate and symmetric.

Proof: Concerning existence, let $\langle \, , \, \rangle$ denote a bilinear form on V from Theorem 9.18. Let $\zeta : V \to V$ denote an isomorphism of $\boxtimes$-modules from V to V twisted via σ. Define a bilinear form $\langle \, , \, \rangle_\sigma$ on V such that $\langle u, v \rangle_\sigma = \langle u, \zeta(v) \rangle$ for all $u, v \in V$. One checks that $\langle \, , \, \rangle_\sigma$ is nonzero and satisfies (61). Concerning uniqueness, let $\langle \, , \, \rangle'_\sigma$ denote a bilinear form on V that satisfies (61). We show that $\langle \, , \, \rangle'_\sigma$ is a scalar multiple of $\langle \, , \, \rangle_\sigma$. Define a bilinear form $\langle \, , \, \rangle'$ on V by $\langle u, v \rangle' = \langle u, \zeta^{-1}(v) \rangle'_\sigma$ for all $u, v \in V$. Then $\langle \, , \, \rangle'$ satisfies (57). Now by the uniqueness of $\langle \, , \, \rangle$ there exists $\beta \in \mathbb{F}$ such that $\langle \, , \, \rangle' = \beta \langle \, , \, \rangle$. This implies that $\langle \, , \, \rangle'_\sigma = \beta \langle \, , \, \rangle_\sigma$. The form $\langle \, , \, \rangle_\sigma$ is nondegenerate since the subspace $\{u \in V | \langle u, v \rangle_\sigma = 0 \ \forall v \in V\}$ is a proper $\boxtimes$-submodule of V and therefore zero by the irreducibility of V. We now show that $\langle \, , \, \rangle_\sigma$ is symmetric. Define a bilinear form $\langle \, , \, \rangle^\dagger_\sigma$ on V by $\langle u, v \rangle^\dagger_\sigma = \langle v, u \rangle_\sigma$ for all $u, v \in V$. By construction and since $\sigma^2 = 1$ we find $\langle \, , \, \rangle^\dagger_\sigma$ satisfies (61). Now by the uniqueness of $\langle \, , \, \rangle_\sigma$ there exists $\gamma \in \mathbb{F}$ such that $\langle \, , \, \rangle^\dagger_\sigma = \gamma \langle \, , \, \rangle_\sigma$. In other words $\langle v, u \rangle_\sigma = \gamma \langle u, v \rangle_\sigma$ for all $u, v \in V$. We show $\gamma = 1$. By Definition 6.8 there exist mutually distinct $i, j, k, \ell \in \mathbb{I}$ such that $\sigma = (i, j)(k, \ell)$. Let $\{V_n\}_{n=0}^d$ denote the decomposition $[i, j]$ of V, and recall that V_0 has dimension 1 by Theorem 9.10. Setting $\xi = x_{ij}$ in (61) and using $\sigma(x_{ij}) = x_{ji} = -x_{ij}$ we find $\langle x_{ij}.u, v \rangle_\sigma = \langle u, x_{ij}.v \rangle_\sigma$ for all $u, v \in V$. It follows that $\{V_n\}_{n=0}^d$ are mutually orthogonal with respect to $\langle \, , \, \rangle_\sigma$. By this and since $\langle \, , \, \rangle_\sigma$ is nondegenerate we find the restriction of $\langle \, , \, \rangle_\sigma$ to V_0 is nonzero. By our above comments, for nonzero $u \in V_0$ we have $\langle u, u \rangle_\sigma \neq 0$ and $\langle u, u \rangle_\sigma = \gamma \langle u, u \rangle_\sigma$ so $\gamma = 1$. We conclude that $\langle \, , \, \rangle_\sigma$ is symmetric. $\square$

At the end of Section 1 we indicated how finite-dimensional irreducible $\boxtimes$-modules give tridiagonal pairs. The bilinear forms in Theorem 9.20 are useful in the theory of these tridiagonal pairs; we will discuss this connection in a future paper.

10 Suggestions for further research

For an integer $N \geq 1$ let $\mathfrak{sl}_2^{(N)}$ denote the Lie algebra $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \cdots \oplus \mathfrak{sl}_2$ (N copies).

Problem 10.1 For an integer $N \geq 1$ and mutually distinct $a_1, a_2, \ldots, a_N$ in $\mathbb{F}\backslash\{0,1\}$, consider the Lie algebra homomorphism $\boxtimes \to \mathfrak{sl}_2^{(N)}$ that sends $\xi \mapsto (EV_{a_1}(\xi), EV_{a_2}(\xi), \ldots, EV_{a_N}(\xi))$ for $\xi \in \boxtimes$. Describe the kernel of this homomorphism in terms of the $\boxtimes$-generators $\{x_{ij} \,|\, i,j \in \mathbb{I}, \, i \neq j\}$ and symmetric functions involving $a_1, a_2, \ldots, a_N$. Also, find an attractive subset of $\boxtimes$ whose image under the homomorphism is a basis for $\mathfrak{sl}_2^{(N)}$.

Example 10.2 With reference to Problem 10.1, assume $N = 2$ and abbreviate $a = a_1$, $b = a_2$. Then the kernel of the homomorphism $\boxtimes \to \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is generated as an ideal by

$$[x_{12}, x_{03}] - 2ab(x_{23} - x_{01}) - 2(1-a)(1-b)(x_{31} + x_{02}),$$
$$[x_{23}, x_{01}] - 2(1-a^{-1})(1-b^{-1})(x_{31} - x_{02}) - 2a^{-1}b^{-1}(x_{12} + x_{03}),$$
$$[x_{31}, x_{02}] - 2(1-a)^{-1}(1-b)^{-1}(x_{12} - x_{03}) - 2ab(1-a)^{-1}(1-b)^{-1}(x_{23} + x_{01}).$$

Also, the images of $x_{12}, x_{23}, x_{31}, x_{01}, x_{02}, x_{03}$ under the homomorphism together form a basis for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

Problem 10.3 Let $U(\boxtimes)$ denote the universal enveloping algebra of $\boxtimes$. For $\sigma \in G$ find a formal sum

$$\Omega_\sigma = \sum_{n=0}^{\infty} t_n \qquad\qquad t_n \in U(\boxtimes)$$

such that the following hold on each finite-dimensional irreducible $\boxtimes$-module V:

(i) t_n is 0 on V for all but finitely many n;

(ii) Ω_σ is invertible on V;

(iii) $\sigma(\xi) - \Omega_\sigma \xi \Omega_\sigma^{-1}$ is 0 on V for all $\xi \in \boxtimes$.

Note 10.4 Let V denote a finite-dimensional irreducible $\boxtimes$-module. Pick $\sigma \in G$ and consider the sum Ω_σ from Problem 10.3. Then the map $V \to V$, $v \mapsto \Omega_\sigma.v$ is an isomorphism of $\boxtimes$-modules from V to V twisted via σ.

Problem 10.5 For an integer $N \geq 1$ let V denote a vector space over $\mathbb{F}$ with dimension $2N$. Let U_i ($i \in \mathbb{I}$) denote N-dimensional subspaces of V such that $U_i \cap U_j = 0$ for distinct $i, j \in \mathbb{I}$. Show that there exists a $\boxtimes$-module structure on V such that $(x_{ij} + I)U_i = 0$ and $(x_{ij} - I)U_j = 0$ for distinct $i, j \in \mathbb{I}$.

Problem 10.6 With reference to the $\boxtimes$-module V in Problem 10.5, show that the following are equivalent:

(i) V is a direct sum of irreducible $\boxtimes$-modules;

(ii) there exists a nondegenerate antisymmetric bilinear form $\langle \, , \, \rangle$ on V such that $\langle U_i, U_i \rangle = 0$ for $i \in \mathbb{I}$.

Problem 10.7 With reference to Notation 7.15, assume $\mathbb{F} = \mathbb{C}$ and that $a \in \mathbb{R}$. Further assume that $\langle \eta_r, \eta_s \rangle \in \mathbb{R}$ for all distinct $r, s \in \mathbb{I}$. Pick mutually distinct $i, j, k, \ell \in \mathbb{I}$ and define $V_{\mathbb{R}} = \sum_{n=0}^{d} \mathbb{R} u_n$, where $\{u_n\}_{n=0}^{d}$ is the basis $[i, j, k, \ell]$ of V from Lemma 7.16. By the data in Theorem 7.20 $V_{\mathbb{R}}$ is independent of i, j, k, ℓ. For a nonidentity $\sigma \in G$ consider the restriction of $\langle\, , \,\rangle_\sigma$ to $V_{\mathbb{R}}$. When is this restriction positive definite?

Problem 10.8 Let V denote a finite-dimensional irreducible $\boxtimes$-module. Find all the linear transformations $\psi : V \to V$ such that both

$$[x_{01}, [x_{01}, [x_{01}, \psi]]] = 4[x_{01}, \psi], \qquad [x_{23}, [x_{23}, [x_{23}, \psi]]] = 4[x_{23}, \psi].$$

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References

[1] G. Benkart and P. Terwilliger. The universal central extension of the three-point $\mathfrak{sl}_2$ loop algebra. *Proc. Amer. Math. Soc.* **135** (2007) 1659–1668. `arXiv:math.RA/0512422`.

[2] G. Benkart and P. Terwilliger. The equitable basis for $\mathfrak{sl}_2$. In preparation.

[3] M. Bremner. Generalized affine Kac-Moody Lie algebras over localizations of the polynomial ring in one variable. *Canad. Math. Bull.* **37** (1994) 21–28.

[4] V. Chari. Integrable representations of affine Lie algebras. *Invent. Math.* **85** (1986) 317–335.

[5] V. Chari and A. Pressley. Quantum affine algebras. *Comm. Math. Phys.* **142** (1991) 261–283.

[6] J. Conway. *Functions of one complex variable.* Graduate texts in Mathematics, 11. *Springer-Verlag, New York-Heidelberg, 1973.*

[7] C. Curtis and I. Reiner. *Representation theory of finite groups and associative algebras.* Pure and Applied Mathematics, Vol. XI. *Interscience Publishers, a division of John Wiley & Sons, New York-London 1962.*

[8] E. Date and S. S. Roan. The structure of quotients of the Onsager algebra by closed ideals. *J. Phys. A: Math. Gen.* **33** (2000) 3275–3296.

[9] B. Davies. Onsager's algebra and superintegrability. *J. Phys. A: Math. Gen.* **23** (1990) 2245–2261.

[10] B. Davies. Onsager's algebra and the Dolan-Grady condition in the non-self-dual case. *J. Math. Phys.* **32** (1991) 2945–2950.

[11] L. Dolan and M. Grady. Conserved charges from self-duality. *Phys. Rev. D (3)* **25** (1982) 1587–1604.

[12] A. Elduque. The S_4-action on the tetrahedron algebra. Proc. Roy. Soc. Edinburgh Sect. A; to appear. `arXiv:math.RA/0604218`.

[13] A. Fialowski and M. Schlichenmaier. Global geometric deformations of current algebras as Krichever-Novikov type algebras. *Comm. Math. Phys.* **260** (2005) 579–612. `arXiv:math.QA/0412113`

[14] B. Hartwig. The tetrahedron algebra and its finite-dimensional irreducible modules. *Linear Algebra Appl.* **422** (2007) 219–235. `arXiv:math.RT/0606197`.

[15] B. Hartwig and P. Terwilliger. The Tetrahedron algebra, the Onsager algebra, and the $\mathfrak{sl}_2$ loop algebra. *J. Algebra* **308** (2007) 840–863. `arXiv:math.ph/0511004`.

[16] J. Humphreys. *Introduction to Lie algebras and representation theory. Second printing, revised.* Graduate Texts in Mathematics, 9. *Springer-Verlag, New York-Berlin, 1978.*

[17] J. Humphreys. *Reflection groups and Coxeter groups.* Cambridge studies in advanced mathematics 29, Cambridge University Press, Cambridge, 1990.

[18] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192; `arXiv:math.CO/0406556`.

[19] T. Ito and P. Terwilliger. Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$. *Ramanujan J.* **13** (2007) 39–62. `arXiv:math.QA/0310042`.

[20] T. Ito and P. Terwilliger. Two non-nilpotent linear transformations that satisfy the cubic q-Serre relations. *J. Algebra Appl.* **6** (2007) 477–503. `arXiv:math.QA/050839`.

[21] T. Ito, P. Terwilliger, and C. W. Weng. The quantum algebra $U_q(\mathfrak{sl}_2)$ and its equitable presentation. *J. Algebra* **298** (2006) 284–301. `arXiv:math.QA/0507477`.

[22] R. Koekoek and R. F. Swarttouw. *The Askey scheme of hypergeometric orthogonal polyomials and its q-analog*, report 98-17, Delft University of Technology, The Netherlands, 1998. Available at `http://aw.twi.tudelft.nl/~koekoek/research.html`

[23] A. Nishino and T. Deguchi. The $L(\mathfrak{sl}_2)$ symmetry of the Bazhanov-Stroganov model associated with the superintegrable chiral Potts model. *Phys. Lett. A* **356** (2006) 366–370.

[24] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)* **65** (1944) 117–149.

[25] A. A. Pascasio and P. Terwilliger. The Tetrahedron algebra and the Hamming graphs. In preparation.

[26] J. H. H. Perk. Star-triangle relations, quantum Lax pairs, and higher genus curves. *Proceedings of Symposia in Pure Mathematics* **49** 341–354. Amer. Math. Soc., Providence, RI, 1989.

[27] M. Schlichenmaier. Higher genus affine algebras of Krichever-Novikov type. *Moscow Math.J.* **3** (2003) 1395–1427.

[28] M. Schlichenmaier. Local cocyles and central extensions for multipoint algebras of Krichever-Novikov type. *J. Reine Angew. Math.* **559** (2003) 53–94.

[29] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* **330** (2001) 149–203; `arXiv:math.RA/0406555`.

[30] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other; comments on the parameter array. *Des. Codes Cryptogr.* **34** (2005) 307–332; `arXiv:math.RA/0306291`.

[31] P. Terwilliger. Leonard pairs and the q-Racah polynomials. *Linear Algebra Appl.* **387** (2004) 235–276; `arXiv:math.QA/0306301`.

[32] P. Terwilliger and R. Vidunas. Leonard pairs and the Askey-Wilson relations. *J. Algebra Appl.* **3** (2004) 411–426; `arXiv:math.QA/0305356`.

[33] P. Terwilliger. The equitable presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$. *J. Algebra* **298** (2006) 302–319. `arXiv:math.QA/0507478`.

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